Dynamics of Surface Roughening with Quenched Disorder

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We study the dynamical exponent z for the directed percolation depinning (DPD) class of models for surface roughening in the presence of quenched disorder. We argue that z for $d + 1$ dimensions is equal to the exponent $d_{\min}$ characterizing the shortest path between two sites in an isotropic percolation cluster in d dimensions. To test the argument, we perform simulations and calculate z for DPD, and $d_{\min}$ for percolation, from $d = 1$ to $d = 6$.

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Recently the growth of rough interfaces has been the object of many theoretical, numerical, and experimental studies, fueled by the broad interdisciplinary aspects of the subject [1]. Applications are as diverse as imbibition in porous media, fluid-fluid displacement, bacterial colony growth, fire front motion, and the motion of flux lines in superconductors [1].

In general, a d-dimensional self-affine interface, described by a single-valued function $h(\mathbf{x}, t)$, evolves in a $(d + 1)$-dimensional medium. Some form of disorder, η, affects the motion of the interface leading to its roughening. Two main classes of disorder have been discussed in the literature. The first, called "annealed," depends only on time. The second, "quenched" disorder, is frozen in the medium.

Continuum equations, such as the Kardar-Parisi-Zhang (KPZ) equation [2], have been remarkably successful in describing roughening for the case of annealed disorder [1]. For the quenched disorder case, several models were proposed with a view toward explaining experimental results for which the roughness exponent α is significantly larger than the predictions for annealed disorder (for reviews, see, e.g., [1]). Here $W \sim L^{\alpha}$, where W is the interface width and L is the system size.

For one class of models, the static properties of the interface in $1 + 1$ dimensions are obtained exactly by a mapping, at the depinning transition, onto *directed percolation* (DP) [3,4]. In higher dimensions the mapping is to *directed surfaces* (DS) [5]—for $1 + 1$ dimensions, DP and DS are equivalent. This class of models is referred to as *directed percolation depinning* (DPD) [6].

A recent numerical study [7], confirmed by scaling arguments [8], showed that the DPD class of models can be described by a stochastic differential equation of the KPZ type with quenched disorder [9]

$$\frac{\partial}{\partial t} h(\mathbf{x}, t) = F + \nu \nabla^2 h + \lambda (\nabla h)^2 + \eta(\mathbf{x}, h) . \quad (1)$$

Here F is the driving force and $\eta(\mathbf{x}, h)$ is the quenched disorder acting to roughen the interface; above the depinning transition, $F \geq F_c$ leads to an interface that moves indefinitely with a constant velocity. Reference [7] finds for the DPD class of models that the coefficient of the nonlinear term, λ, diverges at the depinning transition [10].

Because of the presence of a diverging nonlinear coefficient in (1), the application of the functional renormalization group to the calculation of the exponents has not been possible [1]. For this reason, the only existing estimates of the exponents are from simulations, or from the mapping of the scaling properties of the *pinned interface* to DP ($d = 1$) or DS ($d > 1$) [3–5]. Since no mapping of the *dynamics* of Eq. (1) or of the DPD models has been found, no theoretical estimates for the exponents characterizing the dynamics of the roughening process has been made. One such exponent is the dynamical exponent z, which characterizes the lateral propagation of the interface perturbations, as is defined by

$$t \sim r_{\parallel}^{z} , \quad (2)$$

where t is the time needed for a perturbation to spread over a longitudinal distance $r_{\parallel}$.

In this Letter we argue that z for the DPD universality class can be identified in $d + 1$ dimensions with the percolation exponent $d_{\min}$ for the shortest path for *isotropic* percolation in d dimensions [11]. We support this relation by numerical calculations of both z and $d_{\min}$, up to dimension $6 + 1$. Our work implies an upper critical dimension for the *dynamics*, $d_c + 1 = 7$, above which the mean field result, $z = 2$, becomes exact.

In the DPD model the growth occurs on a discrete lattice, and the disorder is modeled by considering that each cell has a probability p of being "blocked" [3–5]. Since the model was developed to study imbibition, we will refer to the growing, invading, region as "wet," and to the remaining region as "dry." At time $t = 0$, we wet all cells at the bottom of the lattice. Then, at each time step, we wet all dry *unblocked cells* that are nearest neighbors to a wet cell. To retain a single-valued interface, we impose the auxiliary rule that all dry *blocked cells* below a wet cell become wet as well. These cells we call "eroded blocked cells," and this procedure is referred to as *erosion of overhangs* [3–5]. If the concentration of blocked cells is small, the interface propagates forever. As the concentration of blocked cells increases, large portions of

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the interface become pinned by fragments of DP clusters in $d = 1$ or by fragments of DS for $d > 1$. The characteristic longitudinal dimension of these fragments, $\xi_\parallel$, diverges as p approaches the critical threshold p_c. When $\xi_\parallel$ becomes comparable with the system size L, the interface eventually becomes completely pinned by a spanning DP path or DS. Just below p_c, almost all of the interface is pinned except for a few unblocked points which move along the interface creating new sites for growth. We address the behavior of the system only in its critical state, i.e., on length scales smaller than $\xi_\parallel$, where $\xi_\parallel \sim (p_c - p)^{-\nu_\parallel}$.

In order to find z, we study how perturbations caused by a single unblocked cell propagate over the interface. At each time step a certain set of cells become wet by incoming fluid, caused by a single unblocked cell at time $t = 0$. In analogy with invasion percolation, we call this set of cells the *percolation shell*. For each time step t, we compute the average radius of gyration of the percolation shell $r(t)$. Since t is the time needed for a perturbation to spread over a distance $r_\parallel$, t obeys Eq. (2).

For the case $d = 1$, all shells are confined between the old directed path that spans the system at $t = 0$ and a new pinning path that will block the growth after some time. The region between these two paths is topologically one dimensional, since the vertical distance between them scales as the perpendicular correlation length of DP, $\xi_\perp \sim (p_c - p)^{-\nu_\perp}$. Hence $\nu_\perp < \nu_\parallel$ implies $\xi_\perp/\xi_\parallel \to 0$ as $p \to p_c$ [11]. For any cell on the interface that becomes wet at time t, one can find the cell from which it was invaded at the previous time step, and recreate the sequence of invasion events that leads from the initial cell to any given cell on the interface (Fig. 1). The trajectory of this sequence follows the upper pinning path and is effectively one dimensional. Its length ℓ scales as its average end-to-end distance $r_\parallel$. On the other hand, ℓ is equal to the time t needed to reach the end of the path. Hence $t \sim r_\parallel$ and we conclude from (2) that $z = 1$. This conclusion is supported by our simulations (Table I).

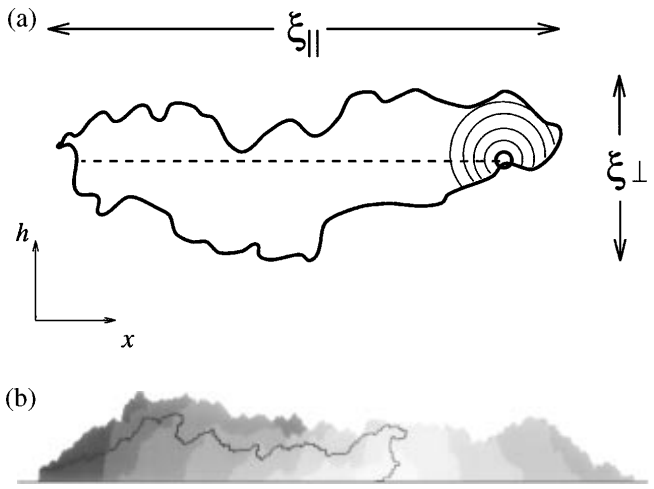

FIG. 1. Illustration of the dynamics of the DPD model for 1 + 1 dimensions. (a) *Schematic representation* of a region defined by two pinning paths. The heavy circle indicates the origin for the invasion, the thin arcs represent the positions of the invading front at successive times, and the dashed line represents schematically the path for the invasion. (b) *Simulation* results for invasion after 2^{10} time steps starting from a single cell near the center. We show the invaded region at a sequence of times which are multiples of 128. Regions invaded at later times are displayed in darker shades of gray. The path from the origin to the latest invaded point is shown in black. Although this path displays some fluctuations in the vertical direction, they can be disregarded since $\nu_\parallel > \nu_\perp$, so as $p \to p_c$, $\xi_\perp/\xi_\parallel \to 0$. Thus the distance propagated by the invading front is proportional to time. Since $t_\times \sim \ell$, we can conclude that $z = d_{\min} = 1$.

For the case $d > 1$, we must consider the region bounded by two self-affine, single-valued, DS (Fig. 2) [5]. This region is topologically d dimensional, since $\xi_\perp/\xi_\parallel \to 0$ [12]. Hence, the shortest path leading from the initial point to any point of this region is effectively confined to a d-dimensional horizontal hyperplane. This shortest path has to avoid blocked cells in this hyperplane,

TABLE I. Dynamical exponent z for the DPD model in $d + 1$ dimensions and the shortest path exponent $d_{\min}$ for isotropic percolation for a d-dimensional cubic lattice of L^d sites. The results indicated by an asterisk are exact, while the remaining values were calculated in our simulations by the study of the consecutive slopes of the linear regime in Fig. 3. At the critical dimension $d_c = 6$, one should not expect to find the exact result $d_{\min} = 2$ because logarithmic corrections are generally present. The system sizes used in the simulations range from $L = 4096$, for $d = 2$, to $L = 16$, for $d = 6$. Each result is averaged over 10^6–10^7 realizations of the disorder.

	DPD		Percolation	
d	p_c	z	p_c	$d_{\min}$
1	0.4698 ± 0.0002	1.01 ± 0.02	1*	1*
2	0.7425 ± 0.0002	1.15 ± 0.05	0.5927 ± 0.0002	1.13 ± 0.03
3	0.8425 ± 0.0002	1.36 ± 0.05	0.3116 ± 0.0002	1.38 ± 0.02
4	0.890 ± 0.002	1.58 ± 0.05	0.197 ± 0.002	1.53 ± 0.05
5	0.917 ± 0.003	1.7 ± 0.1	0.141 ± 0.002	1.7 ± 0.1
6	0.931 ± 0.002	1.8 ± 0.2	0.107 ± 0.002	1.8 ± 0.2

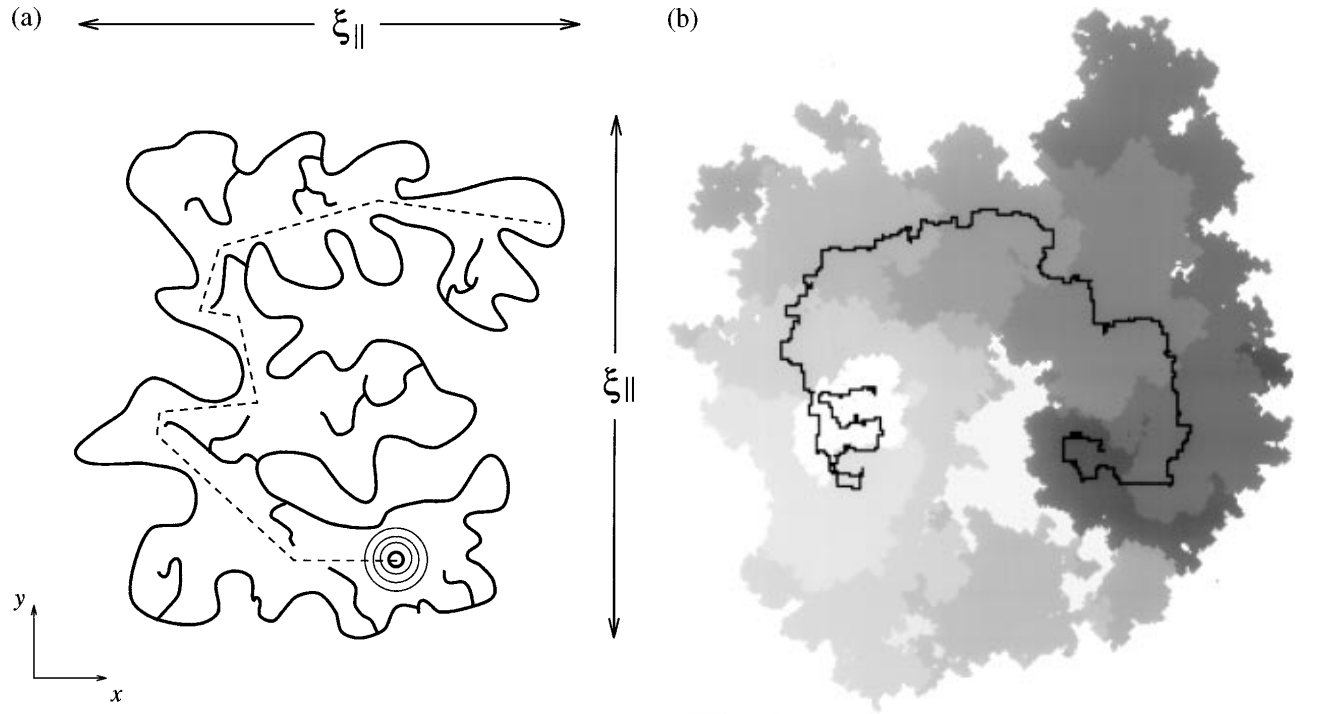

FIG. 2. Illustration of the dynamics of the DPD model for 2 + 1 dimensions. (a) *Schematic representation* of the xy projection of the region defined by two pinning self-affine DS. The heavy circle indicates the origin for the invasion, the thin arcs represent the xy projections of the invading front at successive times, and the dashed line represents schematically the path for the invasion. (b) *Simulation* results for invasion after 2^{10} time steps starting from a single cell located to the left of the center. We show the xy projection of the invaded region at a sequence of times which are multiples of 128. Regions invaded at later times are displayed in darker shades of gray. It is visually apparent that it takes a long time to invade some regions close to the origin because the path to that position (shown in black) appears to be a fractal curve of dimension greater than one. The fluctuations in the vertical direction can be disregarded since we know that $\xi_{\perp}/\xi_{\parallel} \to 0$. We find that the path can be identified with the shortest path (the "chemical distance") of isotropic percolation, and that its length scales with the linear distance r to the point as $r^{d_{\min}}$.

as does the shortest path of isotropic percolation. For isotropic percolation it is known that the length of the shortest path ℓ scales with the Euclidean end-to-end distance r as $\ell \sim r^{d_{\min}}$. The similarity between the geometrical properties of the paths in DPD and isotropic percolation leads us to propose

$$z = d_{\min} . \tag{3}$$

We are arguing that the invading front moves on a d-dimensional isotropic percolation cluster; see Figs. 1 and 2. The critical threshold is smaller for DPD than in the case of the usual isotropic percolation since (i) some of the blocked cells are eroded, and (ii) our system is a d-dimensional slab. The critical threshold can be determined by the spanning of the invading front in the d-dimensional slab. We confirm that we were at the critical threshold by numerically studying the survival probability of these clusters, as described in [5], and verify that we reach the threshold where the invading cluster spans the system.

To test the argument leading to (3), we performed simulations for both DPD and percolation for $d = 1$ to $d = 6$. We present our results for the exponents z and $d_{\min}$ in Table I [13].

It is well known that for isotropic percolation the upper critical dimension is $d_c = 6$, i.e., for $d > d_c$ the mean field result, $d_{\min} = 2$, becomes exact [11]. This suggests an upper critical dimension, $d_c + 1 = 7$, for the *dynamics* of the DPD models which are in the universality class of Eq. (1), and that $z = 2$ for $d + 1 \geq 7$.

Since the dynamics of Eq. (1) and the models in the DPD universality class are connected to isotropic percolation, while the static properties are mapped to DP or DS, it is possible that the upper critical dimension determined in this study *may be valid only for the dynamics*. In fact, it is possible that d_c for the static properties may not exist. Suppose, e.g., that a one-dimensional object, such as a self-avoiding walk, is embedded in a d-dimensional space. We expect that as d is increased the interactions between the different parts of the object decrease. At a certain $d = d_c$, these interactions can be neglected, and the exponents become those of the ideal noninteracting case. In contrast, when the dimension of the object is not fixed but increases with d, as in the case of DS in which the object is one dimension smaller than the space, we expect to move away from the noninteracting limit. In fact, the analytical solution of the DPD model in the Cayley tree suggests that the upper critical dimension for the *statics* might be ∞ [5].

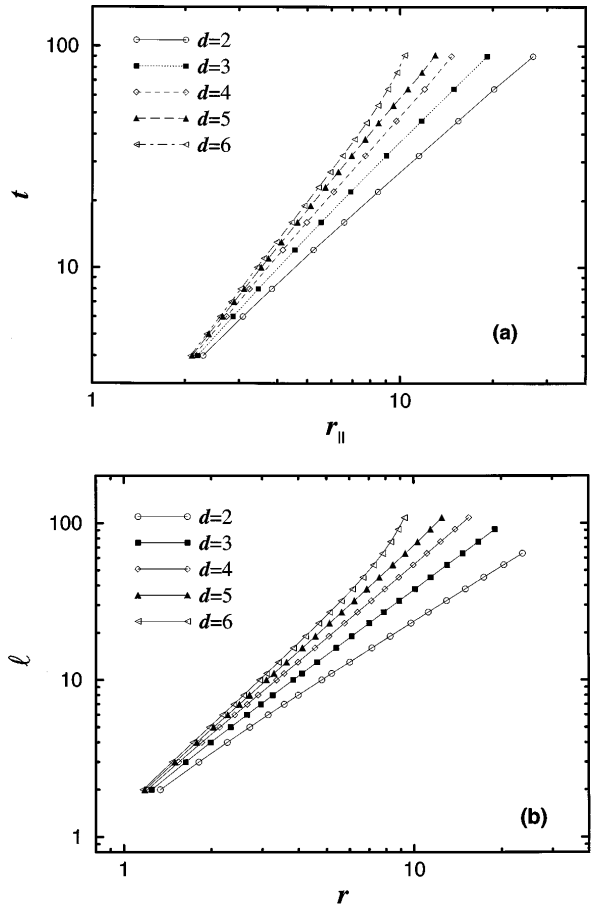

FIG. 3. (a) Scaling with time of the horizontal length of a DPD cluster in $d + 1$ dimensions grown from a single cell. Shown is a double logarithmic plot of time t as a function of $r_{\|}$, which is the average of the parallel components of the radius of gyration of the shell. The asymptotic slope is z. (b) Double logarithmic plot of the shortest path ℓ in isotropic percolation as a function of the Euclidean distance r. The asymptotic slope is $d_{\min}$. Note that, after some transient behavior, a transition to a power law scaling occurs. For higher dimensions, the power law scaling is affected by finite-size effects for larger times [12].

In summary, we present an argument that identifies the dynamical exponent z for the DPD universality class with the fractal dimension of the shortest path in isotropic percolation, $d_{\min}$. This result leads us to identify the dimension $6 + 1$ as the upper critical dimension for the dynamics of the DPD universality class.

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[12] The fact that $\xi_{\perp} \gg 1$, although much smaller than $\xi_{\|}$, leads to a crossover from the $(d + 1)$-dimensional behavior for $t \ll \xi_{\perp}^{z}$ to the d-dimensional behavior for $\xi_{\perp}^{z} \ll t \ll \xi_{\|}^{z}$. The effect of this crossover is most clearly observed in higher dimensions where the simulations cannot access the limit $\xi_{\perp}/\xi_{\|} \to 0$.

[13] The dynamical exponent z discussed here for the DPD class of model is related to the dynamical exponent defined in recent studies of the properties of avalanches in the self-organized depinning (SOD) model [3,14,15]. The differing numerical values of z between the DPD model and the SOD model is due to distinct definitions of time; $t_{\mathrm{SOD}} \sim t^{\delta+1}$, where $\delta = (d + \alpha)/z - 1$ is the number of unblocked active cells defined in [5]. This leads to the relation $z_{\mathrm{SOD}} = z(\delta + 1)$, in good agreement with $z_{\mathrm{SOD}} \simeq 1.63$ found in [15].

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