Energy avalanches in a rice-pile model

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Abstract

We investigate a one-dimensional rice-pile model. We show that the distribution of dissipated potential energy decays as a power law with an exponent $\alpha = 1.53$. The system thus provides a one-dimensional example of self-organized criticality. Different driving conditions are examined in order to allow for comparisons with experiments.

Keywords: Avalanches; Self-organized criticality; Granular systems

1. Introduction

Statistical–mechanical investigations of driven nonequilibrium systems is a field of much current interest. A special class of such systems are those which reach the steady state through a self-organizing dynamics consisting of avalanches propagating through the system. If, in addition, the steady state is characterized by a power-law distribution for the sizes of avalanches, the behavior is referred to as self-organized critical (SOC) [1]. Simple "sandpile" models have been introduced in order to illustrate the SOC behavior [1–8]. Real sandpiles, however, have turned out to display noncritical behavior – i.e., they are described by avalanche size distributions with a characteristic scale [9–14].

Recently, an experiment with rice grains was performed [15]. It was found that a pile of elongated rice grains evolved into a SOC state with the distribution of dissipated potential energy E being flat for small avalanches and crossing over to a power law with exponent $\alpha_{exp} \simeq 2.0$ for large avalanches. Physically motivated rice-pile models have been introduced which indeed do display SOC behavior in one dimension [16–19] but with an exponent which differs from that observed experimentally.

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In this paper, we measure the energy avalanches for the rice-pile model which we introduced in Ref. [16]. Previously, we found that the distribution of avalanches defined as the number of topplings s followed the power-law form $P(s,L) \sim s^{-\tau} f_s(s/L^{v_s})$, with $\tau = 1.53$ and $v_s = 2.20$ [16]. Here, the distribution $P(E,L)$ of energy avalanches in a system of size L is measured. We find that $P(E,L)$ scales with the same exponent as $P(s,L)$ in accordance with the fact that a toppling event on average dissipates a constant energy. We change the drive in our model in order to make comparisons with the experimental results in Ref. [15] more quantitative. Despite, the fact that this leads to a flat distribution for small E, in perfect agreement with the experiment, the power-law tail for large E is still described by an exponent 1.53.

2. Rice-pile model

We briefly describe the model we introduced in [16]. We consider a one-dimensional system of length L with a wall at $i = 0$ and an open boundary at $i = L + 1$, where particles fall off the pile. The dynamics of the model consists of deposition and relaxation: At each time step one grain is added at $i = 1$. Then, the pile is allowed to relax in order to reach a new stable configuration. The relaxation process is considered to be fast compared to the deposition time scale. During the relaxation *active* columns topple one grain from i to $i + 1$ with probability $p(\delta h_i)$, where $\delta h_i \equiv h(i) - h(i + 1)$ is the local slope. A column i of the pile is said to be active if, in the anterior time step, it (i) received a grain from column $i - 1$, (ii) toppled a grain to column $i + 1$, or (iii) column $i + 1$ toppled one grain to its right neighbor. The probability $p(\delta h_i)$ to move a grain is taken to be

$$
p(\delta h_i) = \begin{cases} 0, & \delta h_i \leqslant S_1, \\ p, & S_1 < \delta h_i \leqslant S_2, \\ 1, & S_2 < \delta h_i, \end{cases} \tag{1}
$$

where we will use $S_1 = 1$, $S_2 = 4$, and $p = 0.6$; the results are insensitive to the precise values of these parameters.

Physically, the parameter p describes the friction between rice grains. It also incorporates the possibility that a metastable packing configuration will be reached when a grain topples. The friction effect is the new ingredient in the model compared to other models and it introduces a large range of slopes in the rice pile instead of a single critical value. The parameter S_1 accounts for the fact that small slopes are stable. The parameter S_2 models the effect of gravity on the packing configurations. We assume that above the maximum value S_2 of the local slope, it is no longer possible for a local stable configuration to be achieved, thus a grain must be toppled. In the limiting cases $p = 0, 1$, or $S_2 = S_1$, we recover the model in Ref. [1] (which has trivial behavior in one dimension).

3. Avalanche results

We study the model in the slowly driven limit where the rate of deposition is slow enough that any avalanche, that might be started by a deposited grain, will have ended before a new grain is deposited. The simulations of the model show the existence of a SOC steady state. We follow the definition of Ref. [15] and calculate the size of avalanches as the dissipated potential energy in between snapshots of the profile. Fig. 1(a) shows the probability density of avalanche sizes for different system sizes. We find that the scaling form

$$P(E, L) \sim E^{-\alpha} f(E/L^{\nu})$$ \hfill (2)

describes the distribution of avalanche sizes. The validity of Eq. (2) is reassured by the good data collapse displayed in Fig. 1(b), where we used the values $\alpha = 1.53 \pm 0.05$ and $\nu = 2.20 \pm 0.05$. Since a toppling on average dissipates a constant energy, we expect that $\alpha = \tau$ (and $\nu = \nu_s$) as observed numerically. By assuming that the average value is $\langle E \rangle \sim L$ in the critical state, it follows from Eq. (2) that $\alpha = 2 - 1/\nu$ [20]. This relation is in nice agreement with our numerical values for the exponents. We note that the scaling function f has a peak for values of the argument close to the cutoff region. This is a finite-size effect which is due to the possibility, to form a supercritical state which then relaxes through a very large avalanche [16].

Our estimate for the exponent α is different from the value $\alpha_{\exp} \simeq 2$ reported in Ref. [15], and the scaling function is also quite different from that observed experimentally. Part of the reason for this disagreement is due to the fact that the experimental conditions were different from how we measure the avalanche distribution $P(E, L)$. The next step is thus to study the model under conditions as similar as possible to the experimental ones. In the experiment, grains were deposited randomly in time with a rate of 4–6 grains between successive snapshots of the pile taken every 15 s [15]. Furthermore, since the profile of the pile was not known at every time instant when grains were deposited, the potential energy of the deposited (4–6) grains was estimated from the last profile obtained. The uncertainty in the added potential energy induces a noise level proportional to the system size (see below).

To study the model under the "experimental conditions" we made the following assumptions: (i) snapshots of the profile of the pile are taken every N_s time steps, and (ii) new grains are deposited on the pile at an average rate of $1/N_d$. Thus, every time step there is a probability $1/N_d$ of a new grain being deposited. In our simulations, characteristic values for N_s and N_d were 10^4 and 2100, respectively.

Fig. 2(a) shows the probability density of avalanche sizes for an accurate calculation of E. No flat part is observed for small values of E and no dependence on L is detected – except for finite-size effects. However, when we estimate E as it was done in the experiment of Ref. [15] a significant change occurs. As shown in Fig. 2(b), a plateau whose height depends on L, and originates from the uncertainty in the added potential energy, is observed for small E. For large E, there is a crossover to a power-law behavior at a value that scales with L. In fact, the data resemble quite well the
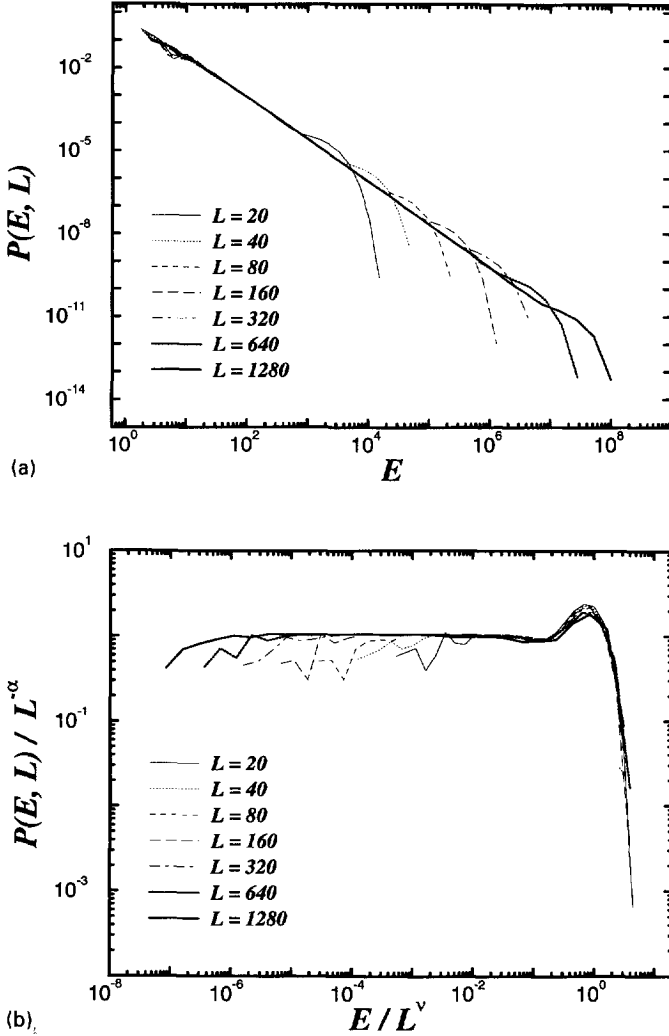
Fig. 1. (a) Log–log plot of the probability density $P(E,L)$ for different values of L. The results were obtained with the parameters $p = 0.6$, $S_1 = 1$, and $S_2 = 4$. The data follows a power-law distribution for several decades with a cutoff that depends on the system size. (b) Data collapse of the curves displayed in (a) according to Eq. (2) with the exponents $\alpha \simeq 1.53$ and $\nu \simeq 2.20$.

experimental results. We find that the data in Fig. 2(b) are well described by the scaling form

$$P(E,L) \sim L^{-\beta}\, g(E/L^{\mu}), \tag{3}$$

where g is a scaling function which is constant for small arguments and decays as a power law for large values of the argument. As can be seen in Fig. 2(c), a good data collapse is obtained with the exponents $\beta = \mu = 1.00 \pm 0.05$ for $L \leqslant 640$. On
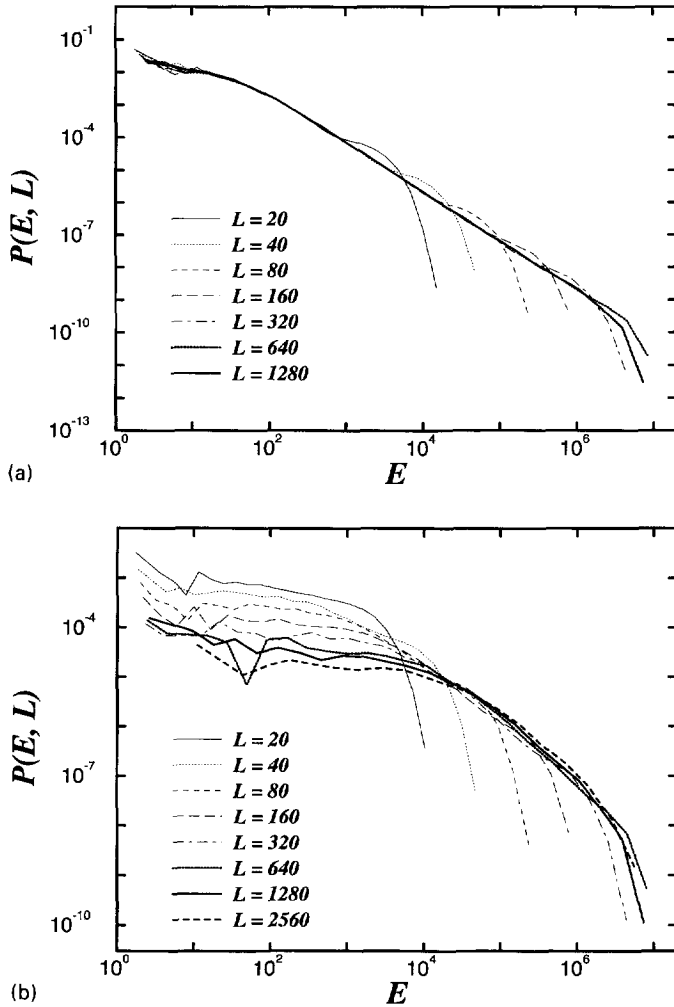
Fig. 2. (a) Log–log plot of the probability density of avalanche sizes for different systems sizes. The data were obtained for conditions similar to the experimental ones but with an accurate determination of the dissipated energy. A power-law dependence is observed with an exponent close to 1.53. (b) Data from the same run as in (a) but now calculating the dissipated energy according to the method used in the experiment. It is remarkable how different the data looks from (a). A plateau for small E is visible and a dependence on L is detectable. (c) Data collapse of the curves shown in (b) according to Eq. (3) with the exponents $\beta = \mu \simeq 1$.

the other hand, for the two larger system sizes we again observe data collapse but to a shifted curve. The reason for this has to do with the possibility of an avalanche running beyond the time at which profiles are measured or a new grain is added. For small L almost every avalanche runs its course before a new grain is deposited or the profile is measured, but for large L we can still have running avalanches before a new picture of the profile is taken, implying that large avalanches are not sampled and the
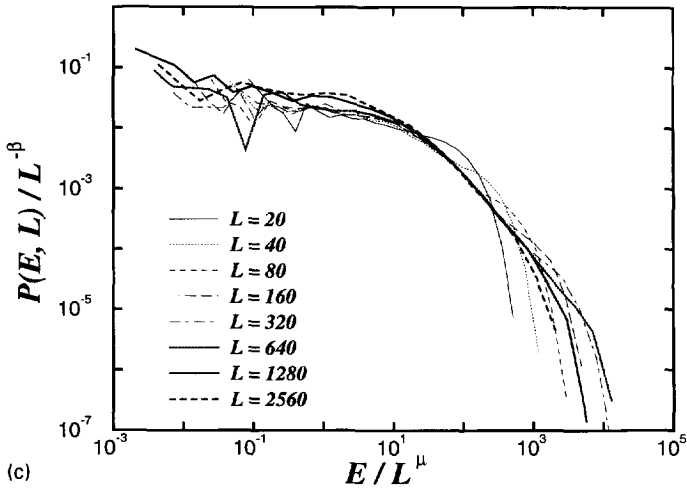
Fig. 2. Continued.

entire distribution is shifted upwards for large energies. This hypothesis is confirmed by the fact that for the largest values of L the size of the biggest avalanche does not seem to grow.

Another observation is that for the inaccurate estimation of the energy, the value of the exponent seems to change from 1.53 to about 1.4. The reason for this can be understood as a simple finite-size effect and that it is only for a small region where the true power-law behavior is observed. One way to circumvent the finite-size effect is to measure instead the slope of the envelope which is close to 1.53. Thus, we expect the correct value of the exponent α to be 1.53.

4. Conclusions

We investigate energy avalanches in a one-dimensional rice-pile model. We find that Eq. (2) provides a good description of our numerical results for the model when we drive it slowly. Furthermore, when we drive the model in a way close to the experimental conditions the numerical results are described by Eq. (3) in nice agreement with the experimental results in Ref. [15]. However, the numerical value for the exponent describing the power-law decay is 1.53, whereas the experiment gave the value $\alpha_{exp} \simeq 2$. Despite the fact that the overall scaling form can be understood by the rice model studied here, the disagreement in the values for α shows that the model needs extensions. One such possibility would be to include the effect of the kinetic energy of the particles in addition to the friction parameter p considered here.

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