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# Scaling behavior in economics: The problem of quantifying company growth

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**Abstract**

Inspired by work of both Widom and Mandelbrot, we analyze the Computstat database comprising all publicly traded United States manufacturing companies in the years 1974–1993. We find that the distribution of company sizes remains stable for the 20 years we study, i.e., the mean value and standard deviation remain approximately constant. We study the distribution of sizes of the "new" companies in each year and find it to be well approximated by a log-normal. We find (i) the distribution of the logarithm of the growth rates, for a fixed growth period of $T$ years, and for companies with approximately the same size $S$ displays an exponential "tent-shaped" form rather than the bell-shaped Gaussian, one would expect for a log-normal distribution, and (ii) the fluctuations in the growth rates – measured by the width of this distribution $\sigma_T$ – decrease with company size and increase with time $T$. We find that for annual growth rates ($T = 1$), $\sigma_T \sim S^{-\beta}$, and that the exponent $\beta$ takes the same value, within the error bars, for several measures of the size of a company. In particular, we obtain $\beta = 0.20 \pm 0.03$ for sales, $\beta = 0.18 \pm 0.03$ for number of employees, $\beta = 0.18 \pm 0.03$ for assets, $\beta = 0.18 \pm 0.03$ for cost of goods sold, and $\beta = 0.20 \pm 0.03$ for property, plant, and equipment. We propose models that may lead to some insight into these phenomena. First, we study a model in which the growth rate of a company is affected by a tendency to retain an "optimal" size. That model leads to an exponential distribution of the logarithm of growth rate in agreement with the empirical results. Then, we study a hierarchical tree-like model of a company that enables us to relate $\beta$ to parameters of a company structure. We find that $\beta = -\ln \Pi / \ln z$, where $z$ defines the mean branching ratio of the hierarchical tree and $\Pi$ is the probability that the lower levels follow the policy of higher levels in the hierarchy. We also study the output distribution of growth rates of this hierarchical model. We find that the distribution is consistent with the exponential form found empirically. We also discuss the time dependence of the shape of the distribution of the growth rates.

*Dedicated to Professor Ben Widom on the occasion of his 70th birthday*



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## 1. Introduction

Ben Widom was the first of a pioneering group of statistical physicists to introduce concepts of scaling in critical phenomena [1]. At approximately the same time, Benoit Mandelbrot was analyzing the distribution of the price changes in cotton at different time scales [2–8]. The Mandelbrot work has inspired many recent studies. In particular the databases that can be analyzed today involve records not at daily intervals but at 1 min (or even quarter minute) intervals [9–27].

Here we discuss not scaling in finance, but rather scaling in economics. Specifically, inspired by both Widom and Mandelbrot, we extend the analysis of the Computstat database comprising all publicly traded United States manufacturing companies within the years 1974–1993 with a view toward characterizing the growth rates of manufacturing companies [28,31].

There are often two motivations for being interested in a given question, a practical and a scientific motivation. The same is true for the distribution in growth rates of companies. A practical motivation is that if you want to invest in a given company, you may wish to know in advance the probability that this company will grow by a given amount, so you need to know the histogram of growth rates – and there is a quite different histogram depending on the size of the firm in question.

A scientific motivation for the present study is the considerable recent interest in economics in developing a richer theory of the company [32–50]. In standard microeconomic theory, a company is viewed as a production function for transforming inputs such as labor, capital, and materials into output [34,39,45]. When dynamics are incorporated into the model, the source of the link between production in one period and production in another arises because of investment in durable, physical capital and because of technological change (which in turn can arise from investments in research and development). Recent work on company dynamics emphasizes the effect of how companies learn over time about their efficiency relative to competitors [38,51–53]. The production dynamics captured in these models are not, however, the only source of actual company dynamics. Most notably, the existing models do not account for the time needed to assemble the organizational infrastructure needed to support the scale of production that typifies modern corporations.

## 2. Background

In 1931, the French economist Gibrat proposed a simple model to explain the empirically observed size distribution of companies [32]. He made the following assumptions:

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(i) the growth rate $R$ of a company is independent of its size (this assumption is usually referred to by economists as the *law of proportionate effect*), (ii) the successive growth rates of a company are uncorrelated in time, and (iii) the companies do not interact.

In mathematical form, Gibrat's model is expressed by the stochastic process:

$$S_{t+\Delta t} = S_t(1 + \varepsilon_t)\,, \tag{2.1}$$

where $S_{t+\Delta t}$ and $S_t$ are, respectively, the size of the company at times $(t + \Delta t)$ and $t$, and $\varepsilon_t$ is an uncorrelated random number with some bounded distribution and variance much smaller than one (usually assumed to be Gaussian). Hence, $\log S_t$ follows a simple random walk and, for sufficiently large time intervals $T \gg \Delta t$, the growth rates

$$R_T \equiv \frac{S_{t+T}}{S_t} \tag{2.2}$$

are log-normally distributed. If we assume that all companies are born at approximately the same time and have approximately the same initial size, then the distribution of company sizes is also log-normal.

An advantage of Gibrat's model is that it yields testable hypotheses. The law of proportionate effect implies that the mean growth rate and the fluctuations of the growth rate are independent of size. In fact, however, the fluctuations of the growth rate measured by the standard deviation $\sigma(S)$ decline with an increase in company size. This was first observed by Singh and Whittington [54] and confirmed by others [28–31,55–59].The negative relationship between growth fluctuations and size is not surprising because large companies are likely to be more diversified. Singh and Whittington state that the decline of the standard deviation with size is not as rapid as if the companies consisted of independently operating subsidiary divisions. The latter would imply that the relative standard deviation decays as $\sigma(S) \sim S^{-1/2}$ [54]. This confirms the common-sense view that the performance of different parts of a company are related to each other.

## 3. Empirical results

We study all the US manufacturing publicly traded companies from 1974 to 1993. The source of our data is Compustat which is a database on all publicly traded companies in the US. Compustat obtains this information from reports that publicly traded companies must file with the US Securities and Exchange Commission. The database contains a large amount of information on each company. Among the items included are "sales", "cost of goods sold", "assets", "number of employees", and "property, plant, and equipment".

Another item provided for each company is the Standard Industrial Classification (SIC) code. In principle, two companies in the same primary SIC code are in the same market; that is, they compete with each other. In practice, defining markets is extremely difficult [53]. More important for our analysis, virtually all modern companies sell in more than one market.

The only use we make of the primary SIC codes in Compustat is to restrict our attention to manufacturing companies. Specifically, we include in our sample all companies with a major SIC code from 2000 to 3999. We do not use the data from the individual business segments of a company, nor do we divide up the sample according to SIC codes.

The approach we take in this study is part of a distinguished tradition. First, there is a large body of work by Simon [40] and various co-authors that explored the stochastic properties of the dynamics of company growth. Also, in a widely cited article (that nonetheless has not had much impact on mainstream economic analysis), Lucas suggests that the distribution of company size depends on the distribution of managerial ability in the economy rather than on the factors that determine size in the conventional theory of the company [41].

To study the distribution of company sizes and growth rates, one problem that must be confronted is the definition of company size. If all companies produced the same good (say, steel), then we could use a physical measure of output, such as tons. We are, however, studying companies that produce different goods for which there is no common physical measure of output. An obvious solution to the problem is to use the dollar value of output: the sales. A general alternative to measuring the size of output is to measure input. Again, since companies produce different goods, they use different inputs. However, virtually all companies have employees. As a result, some economists have used the number of employees as a measure of company size. Three other possibilities involve the dollar value of inputs, such as the "cost of goods sold", "property, plant and equipment", or "assets". As we discuss below, we obtain similar results for all of these measures. We begin by describing the growth rate of sales. To make the values of sales in different years comparable, we adjust all values to 1987 dollars by the GNP price deflator.

Since the law of proportionate effects implies a multiplicative process for the growth of companies, it is natural and more convenient to study the logarithm of sales. We thus define

$$s_0 \equiv \ln S_0 \tag{3.1}$$

and the corresponding growth rate

$$r_1 \equiv \ln R_1 = \ln \frac{S_1}{S_0}\,, \tag{3.2}$$

where $S_0$ is the size of a company in a given year and $S_1$ its size the following year.

### 3.1. Size distribution of publicly traded companies

Stanley et al., determined the size distribution of publicly traded manufacturing companies in the US [60]. They found that for 1993 the data fit to a good degree of

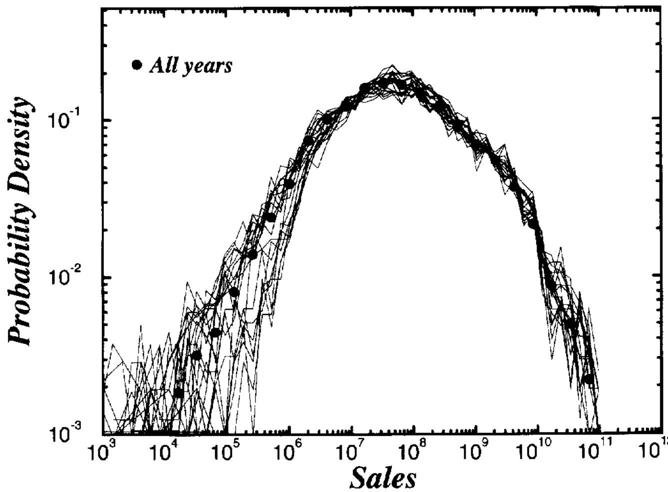



Fig. 1. Probability density of the logarithm of the sales for publicly-traded manufacturing companies (with standard industrial classification index of 2000–3999) in the US for *each* of the years in the 1974–1993 period. All the values for sales were adjust to 1987 dollars by the GNP price deflator. Also shown (solid circles) is the average over the 20 years. It is visually apparent that the distribution is approximately stable over the period.

approximation a log-normal distribution. These results have been recently confirmed by Hart and Oulton [61] for a sample of approximately 80 000 United Kingdom companies. Here, we present a study of the distribution for a period of 20 years (from 1974 to 1993).

Fig. 1a shows the distribution of company size in each year from 1974 to 1993. Particularly above the lower tails, the distributions lie virtually on top of each other. Thus, the distribution is stable over this period. This is a surprising result, when we compare it with the predictions of the Gibrat model. Eq. (2.1) implies that the distribution of sizes of companies should get broader with time. In fact, the variance of the distribution should increase linearly in time. Thus, we must conclude that other factors, not included in Gibrat's assumptions, must have important roles. Two obvious factors not captured by the Gibrat assumption are (i) the entry of new companies and (ii) the "dying" of companies.

### *3.2. The distribution of annual growth rates*

The distribution $p(r_1 \,|\, s_0)$ of the growth rates from 1974 to 1993 is shown in Fig. 2 for three different values of the initial sales [62]. Remarkably, these curves display a simple "tent-shaped" form. Hence, the distribution is not Gaussian – as expected from the Gibrat approach [32] – but rather is exponential [28–30,31]

$$p(r_1 \,|\, s_0) = \frac{1}{\sqrt{2}\sigma_1(s_0)} \exp\left( - \frac{\sqrt{2}\,|r_1 - \bar{r}_1(s_0)|}{\sigma_1(s_0)} \right) . \tag{3.3}$$

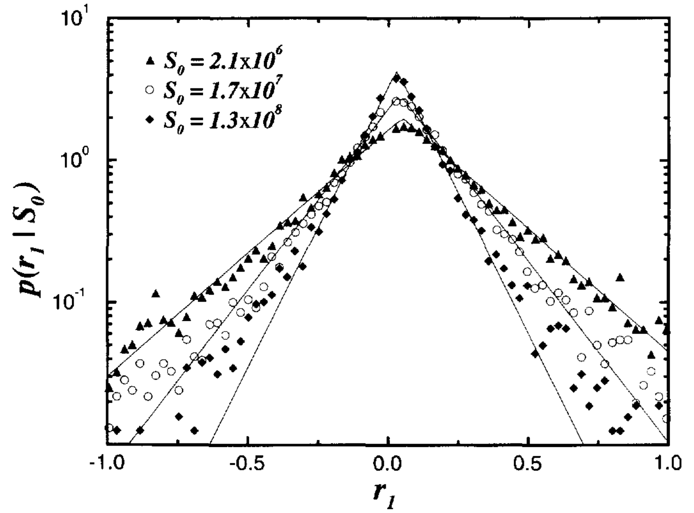



Fig. 2. Probability density $p(r_1|s_0)$ of the growth rate $r \equiv \ln(S_1/S_0)$ for all publicly-traded US manufacturing companies in the 1994 Compustat database with Standard Industrial Classification index of 2000–3999. The distribution represents all annual growth rates observed in the 19-year period 1974–1993. We show the data for three different bins of initial sales (with sizes increasing by powers of 8): $8^7 < S_0 < 8^8$, $8^8 < S_0 < 8^9$, and $8^9 < S_0 < 8^{10}$. The solid lines are exponential fits to the empirical data close to the peak. We can see that the wings are somewhat "fatter" than what is predicted by an exponential dependence.

The straight lines shown in Fig. 2 are calculated from the average growth rate $\bar{r}_1(s_0)$ and the standard deviation $\sigma_1(s_0)$ obtained by fitting the data to Eq. (3.3). An implication of this result is that the distribution of the growth rate has much broader tails than would be expected for a Gaussian distribution.

### 3.3. Standard deviation of the growth rate

Next, we study the dependence of $\sigma_1(s_0)$ on $s_0$. As is apparent from Fig. 2, the width of the distribution of growth rates decreases with increasing $s_0$. We find that $\sigma_1(s_0)$ is well approximated for eight orders of magnitude (from sales of less than $10^3$ dollars up to sales of more than $10^{11}$ dollars) by the law [28–30,31]

$$\sigma_1(s_0) \sim \exp(-\beta s_0)\,, \tag{3.4}$$

where $\beta = 0.20 \pm 0.03$. Eq. (3.4) implies the scaling law

$$\sigma_1(S_0) \sim S_0^{-\beta}\,. \tag{3.5}$$

Fig. 3 displays $\sigma_1$ vs. $S_0$, and we can see that Eq. (3.5) is indeed verified by the data.

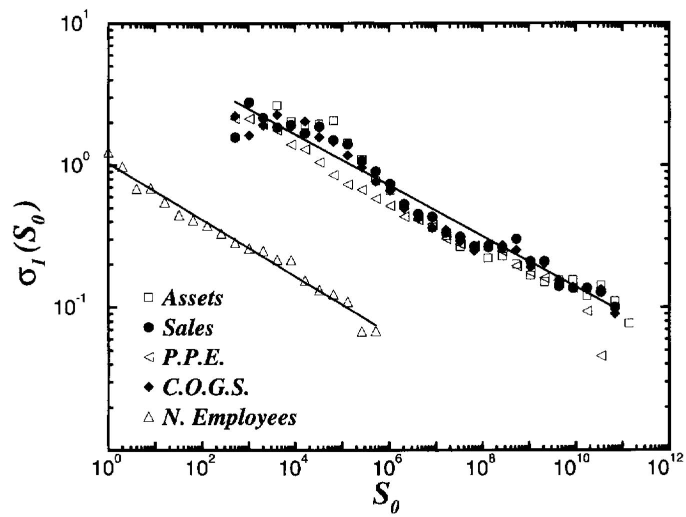



Fig. 3. Standard deviation of the 1-year growth rates for different definitions of the size of a company as a function of the initial values. Least-squares power-law fits were made for all quantities leading to the estimates of $\beta$: $0.18 \pm 0.03$ for "assets", $0.20 \pm 0.03$ for "sales", $0.18 \pm 0.03$ for "number of employees", $0.18 \pm 0.03$ for "cost of goods sold", and $0.20 \pm 0.03$ for "plant, property & equipment". The straight lines are guides to the eye and have slopes 0.19.

Also of interest is the width of the distribution of final sizes $S_1 \equiv S_0 \exp r_1$, that we designate by $\Sigma_1(S_0)$. We can express $\Sigma_1$ as

$$\Sigma_1(S_0)^2 = \langle S_1^2 \rangle - \langle S_1 \rangle^2 \tag{3.6}$$

which scales as

$$\Sigma_1(S_0) \sim S_0^{1-\beta} . \tag{3.7}$$

### 3.4. The T-year growth rates

Another relevant question is the validity of Eq. (3.3) for larger periods of time, i.e., if we consider the $T$-year growth rate $r_T$, will we get a similar distribution or not? The analysis of the data shows that the distribution of growth rates for $T$ as large as 8 yr does not follow a log-normal distribution.

We find that for $T \leqslant 8$ the distribution of growth rates approximately follows an exponential distribution; cf. Fig. 4a. For $T = 16$ the results are not clear due to the noise.

Finally, we study the dependence of the width of the distribution, for a given value of $s_0$, on time. Fig. 4b suggests that $\sigma_T(s_0)$ grows as a logarithm or a small power of $T$.

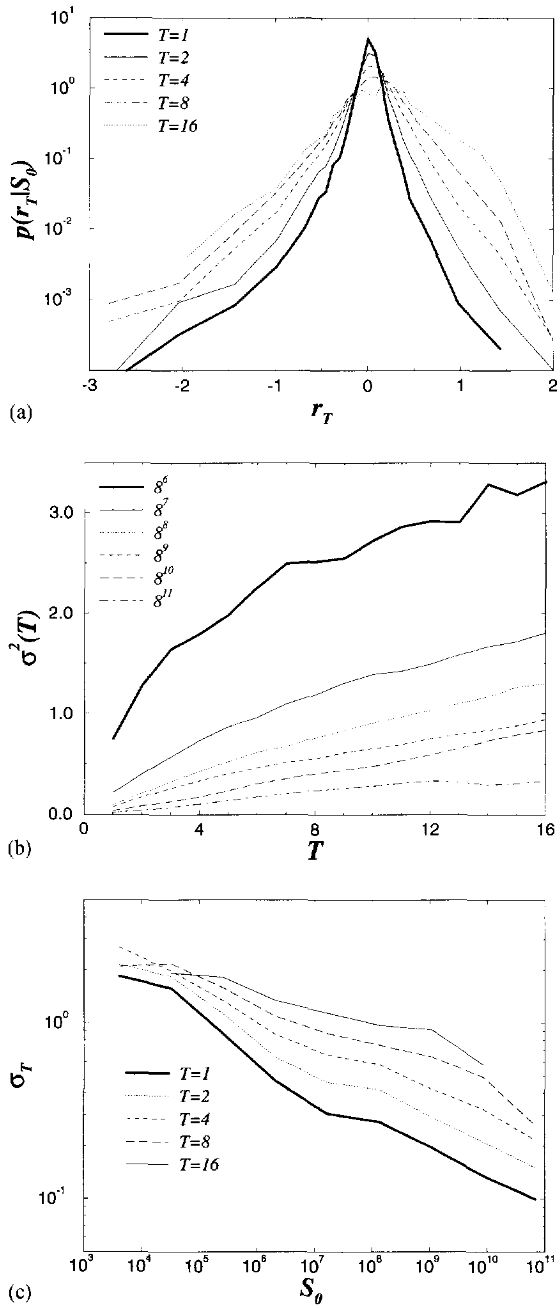



Fig. 4. (a) Probability density of the $T$-year growth rate for companies with initial size of $8^8 < S_0 < 8^9$. It is visually apparent that, at least for $T \leqslant 8$, the distribution is well approximated in its central part by Eq. (3.3). (b) Plot of the average square width of the distribution $\sigma_T^2$ as a function of $T$ for different values of $S_0$. It is clear that $\sigma^2$ increases slower than linearly. This result implies anti-correlations in the successive one-year growth rates. (c) Plot of the average width of the distribution $\sigma_T$, as a function of $S_0$ for different values of $T$. It is clear that size dependence of $\sigma_T$ becomes weaker for larger values of $S_0 T$.

For large company sizes the growth of $\sigma_T$ can, to some degree, be approximated by $\sqrt{T}$, which is expected for independent successive annual growth rates. However, for small companies, $\sigma_T$ grows more slowly than $\sqrt{T}$, thus, suggesting that 1 yr growth rates are anticorrelated. Our data also suggest that the exponent $\beta$ is not universal but decreases with $T$ (see Fig. 4c).

### 3.5. Discussion

What is remarkable about Eqs. (3.3) and (3.5) is that they approximate the growth rates of a diverse set of companies. They differ not only in their size but also in what they manufacture. The conventional economic theory of the company is based on production technology, which varies from product to product. Conventional theory does not suggest that the processes governing the growth rate of car companies should be the same as those governing, e.g., pharmaceutical or paper companies. Indeed, our findings are reminiscent of the concept of universality found in statistical physics, where different systems can be characterized by the same fundamental laws, independent of "microscopic" details. Thus, we can pose the question of the universality of our results: Is the measured value of the exponent $\beta$ due to some averaging over the different industries, or is it due to a universal behavior valid across all industries? As a "robustness check", we split the entire sample into two distinct intervals of SIC codes. It is visually apparent in Fig. 5a that the same behavior holds for the different industries. Thus, we can conclude that our results are indeed universal across different manufacturing industries in the US.

In statistical physics, scaling phenomena of the sort that we have uncovered in the sales and employee distribution functions are sometimes represented graphically by plotting a suitably "scaled" dependent variable as a function of a suitably "scaled" independent variable. If scaling holds, then the data for a wide range of parameter values are said to "collapse" upon a *single* curve. To test the present data for such data collapse, we plot in Fig. 5b the scaled probability density $p_{\rm scal} \equiv \sqrt{2}\sigma(s_0)\, p(r_1 \mid s_0)$ as a function of the scaled growth rates of both sales and employees $r_{\rm scal} \equiv \sqrt{2}[r_1 - \bar{r}_1(s_0)]/\sigma(s_0)$. The data collapse upon the single straight line $p_{\rm scal} = \exp(-|r_{\rm scal}|)$shows small but consistent deviations for large growth rates from the exponential distribution in Eq. (5). Thus, Eq. (5) can be regarded only as a first-order approximation to reality. Our results for (i) cost of goods sold, (ii) assets, and (iii) property, plant and equipment are equally consistent with such scaling. Fig. 5c represents the analogous plot for growth rates for different time periods $T$. It can be seen that the shape of the distribution remains practically unchanged for larger periods of time $T > 1$. Regardless of the exact validity of Eqs. (5) and (7), it is remarkable that the shape of the distribution is similar for all company sizes and does not converge to a Gaussian, even for large $T$ – as the Gibrat model [Eq. (1)] would predict.

The high degree of similarity in the behavior of sales, the number of employees, and of the other measures of size that we studied points to the existence of large correlations among those quantities, as one would expect.

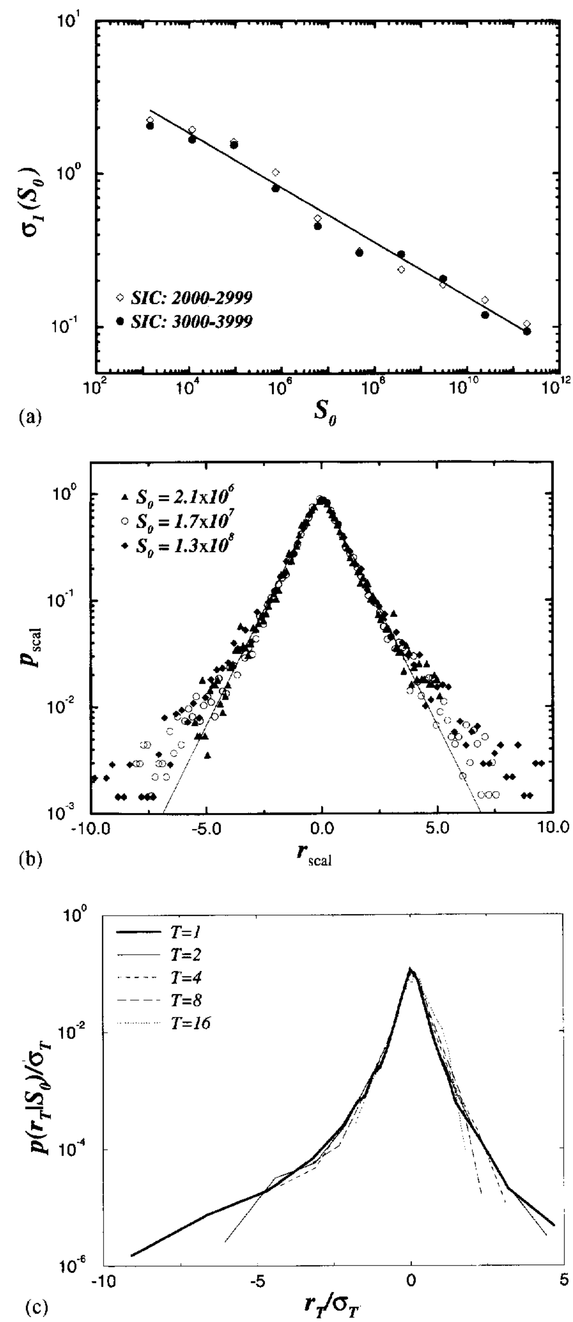



Fig. 5. (a) Dependence of $\sigma_1$ on $S_0$ for two subsets of the data corresponding to different values of the SIC codes. In principle, companies in different subsets operate in different markets. The figure suggests that our results are universal across markets. (b) Scaled probability density $p_{\mathrm{scal}} \equiv \sqrt{2}\sigma_1(s_0)p(r_1|s_0)$ as a function of the scaled growth rate $r_{\mathrm{scal}} \equiv \sqrt{2}[r_1 - \bar{r}_1(s_0)]/\sigma_1(s_0)$. The values were rescaled using the measured values of $\bar{r}_1(s_0)$ and $\sigma_1(s_0)$. All the data collapse upon the universal curve $p_{\mathrm{scal}} = \exp(-|r_{\mathrm{scal}}|)$ as predicted by Eqs. (3.3) and (3.4). (c) Similar scaling plot for the data from Fig. 4(a). Again, we can see that all the data collapse onto a single curve.

## 4. Stochastic modeling

In this section we will present and discuss models that, although very simple, may give some insight into the empirical results. First, we look into the problem of the distribution of growth rates. The generally weak assumptions underlying the central limit theorem suggests that the distribution would be Gaussian. In fact, however, the data have an exponential distribution not only for $r_1$ but also for $r_2$, $r_4$, and $r_8$.

A second puzzle is the striking simplicity of the power-law dependence of $\sigma_1$ on $S_0$. Such a result is reminiscent of critical phenomena and hints at the possibility of the economy self-organizing into a critical state [70].

### *4.1. The exponential distribution of growth rates*

The central limit theorem suggests that the distribution of $T$-year growth rates should be a Gaussian for $T$ sufficiently large. However, the analysis of the data shows that Eq. (3.3) is verified for $T \leqslant 8$, while for $T = 16$, the noise makes any interpretation difficult.

Thus, we can ask if there is a plausible modification of Gibrat's assumptions [32] that could lead to Eq. (3.3). One possibility is to relax the assumption of uncorrelated growth rates and to assume that the successive growth rates are correlated in such a way that the size of a company is "attracted" to an optimal size $S^*$. This value may be interpreted as the minimum point of a "U-shaped" average cost curve in conventional economic theory and should evolve only slowly in time (on the scale of years) [63]. Let us then consider a set of companies all having initial sales $S_0$. As time passes, the sales of each of the companies will vary from day to day (or over another time interval much less than 1 yr), but they tend to stay in the neighborhood of $S^*$. In the simplest case, the growth process has a constant "back-drift," i.e.

$$\frac{S_{t+\Delta t}}{S_t} = \begin{cases} k \exp(\varepsilon_t), & S_t < S^*, \\ \frac{1}{k} \exp(\varepsilon_t), & S_t > S^*, \end{cases} \tag{4.1}$$

where $k$ is a constant larger than one and $\varepsilon_t$ an uncorrelated Gaussian random number with zero mean and variance $\sigma_\varepsilon^2 \ll 1$. These dynamics are similar to what is known in economics as regression towards the mean [64,65], although this formulation is not standard in economics.

This is a well-known problem [66], and, for large times $t$, $r$ is distributed according to the equilibrium Boltzmann distribution,

$$p(r_1 \mid s_0) = \frac{\ln k}{\sigma_\varepsilon^2} \exp\left( -\frac{2 \ln k \mid r_1 - r^* \mid}{\sigma_\varepsilon^2} \right) . \tag{4.2}$$

Hence, we recover Eq. (3.3) with $\bar{r}(s_0) = r^*$ and

$$\sigma_1(s_0) = \frac{\sigma_\varepsilon^2}{\sqrt{2} \ln k} . \tag{4.3}$$

The results expressed by Eqs. (4.2) and (4.3) can account for the increase of $\sigma_1$ with the size of the company if we assume that $\sigma_\varepsilon$ is a function of $s_0$. A model for such a dependence will be discussed in Section 4.3.

### 4.2. *Time dependence of the growth-rate distribution*

Eq. (4.2) describes the equilibrium distribution of the growth rates for sufficiently long times $t$. Our data suggest that $\sigma_t$ grows with time, even for $t = 16$. One possible explanation is that we are still in the transient regime of the process in Eq. (4.1). In order to find the distribution in the transient regime, we must write down the Fokker–Planck equation [66] associated with this process:

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\sigma_\varepsilon^2}{\Delta t}\frac{\partial^2 f}{\partial r^2} + \frac{\ln k}{\Delta t}\frac{\partial f}{\partial r}\,\mathrm{sign}(r - r^*)\,. \tag{4.4}$$

Using dimensionless variables

$$x = \frac{r - r^*}{r_0}, \qquad u = \frac{t}{t_0}\,, \tag{4.5}$$

where

$$r_0 = \frac{\sigma_\varepsilon^2}{\ln k} \quad \text{and} \quad t_0 = \frac{\Delta t \sigma_\varepsilon^2}{(\ln k)^2}\,, \tag{4.6}$$

and imposing a mass-conservation condition

$$\int_0^\infty p(x)\,dx = \int_{-\infty}^0 p(x)\,dx = \frac{1}{2}\,,$$

we get the solution

$$p(x,u) = \frac{1}{\sqrt{2\pi u}} e^{-(|x|+u)^2/2u} + \frac{1}{2}\,\mathrm{erfc}\left(\frac{|x| - u}{\sqrt{2u}}\right) e^{-2|x|}\,, \tag{4.7}$$

which always satisfies the boundary condition

$$\left.\frac{\partial \ln p}{\partial x}\right|_{x=+0} = -2\,.$$

For large $u \gg x$, $u \gg 1\,(t \gg r\,\Delta t/\ln k,\ t \gg \Delta t\sigma_\varepsilon^2/(\ln k)^2)$, in agreement with Eq. (4.2), the distribution can be well approximated by an exponential form:

$$p(r\,|\,s_0) = \frac{1}{r_0} e^{-|r - r^*|/r_0}\,. \tag{4.8}$$

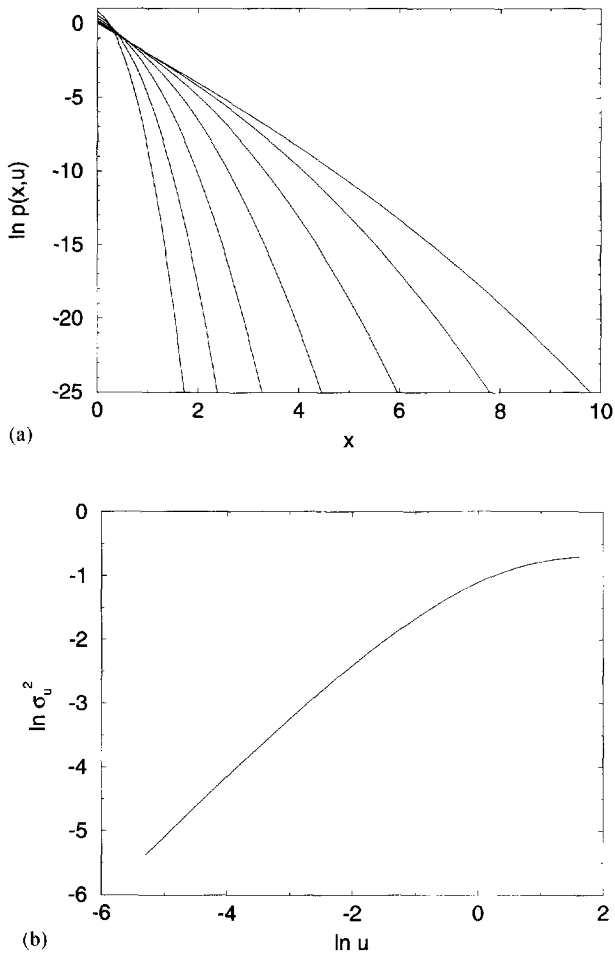



Fig. 6. (a) Solutions of Eq. (4.4), given by Eq. (4.7), for several values of $u$ (from right to left, $u = 4, 2, 1, 1/2, 1/4, 1/8, 1/16$). (b) The width of the distribution (4.7) $\sigma_u^2$ given by Eq. (4.9).

For small $u \ll x$ the slopes of the graphs of the ln $p(x,u)$ can be well approximated by a linear equation $\partial \ln p/\partial x \approx -1 - x/u$, and thus the distributions $p(x,u)$ for large $x$ are parabolas widening with the increase of $u$ [see Fig. 6a]. The width of the distribution $p(x,u)$ is given by

$$\sigma_u^2 = \int_{-\infty}^{+\infty} p(x,u) x^2 \, dx$$
$$= \frac{1}{2} \operatorname{erf}\sqrt{\frac{u}{2}} + \left(u + \frac{1}{2}u^2\right) \operatorname{erfc}\sqrt{\frac{u}{2}} - \frac{1}{\sqrt{2\pi u}}(u^2 + u)\,\mathrm{e}^{-u/2} \,. \tag{4.9}$$

For small $u \ll 1$, $\sigma_u^2$ increases linearly with time, but for large $u$, converges to its limiting value $\frac{1}{2}$ in agreement with Eq. (4.3) [see Fig. 6b]. In terms of the original variable $t$, it happens when $t \gg t_0 = (\Delta t \sigma_\varepsilon^2)/(\ln k)^2$. The comparison of our experimental data with Eqs. (4.7) and (4.9) suggest that these two equations correctly predict the qualitative behavior of $p(r_t, s_0)$ and $\sigma_t$, but fail to reproduce important quantitative details of the experimental data.

First, the distribution (4.7) for large $x$ has a rate of decay faster than exponential while the real data have a rate of decay slower than exponential. Second, the distribution (4.7) always has a slope of $-2$ near the peak, while the slopes of the real graphs apparently decrease with time. Finally, the behavior of $\sigma_t$ (4.9) has a sharp crossover at time $t_0$ from linear growth to constant, while the real data can be approximated as weak power law for long time spans. This means that for real data the transient time $t_0$ is very large.

Another possible explanation for the time-dependence of $\sigma_T$ is that the optimal size of a company does not remain constant but, in fact, performs some sort of random walk with a very small diffusion coefficient $\mathscr{D}$. Such a model can be easily solved and it leads to the prediction that

$$p(s_T \mid s_0) = \frac{1}{2a} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-(s^* - s_0)^2/2t} \, e^{-(s_T - s^*)/a} \, ds^* \tag{4.10}$$

where $a = \sigma_1(s_0)/\sqrt{2}$ and $2t = \mathscr{D}T$. The analytical form of the distribution of growth rates is then given by

$$p(r_T \mid s_0) = \frac{1}{2a} e^{T/2a^2} \left[ e^{-r_T/a} \operatorname{erfc}\left( \frac{t/a - r_T}{\sqrt{2t}} \right) + e^{r_T/a} \operatorname{erfc}\left( \frac{t/a + r_T}{\sqrt{2t}} \right) \right] , \tag{4.11}$$

where $\operatorname{erfc} x = 2/\sqrt{\pi} \int_x^{+\infty} \exp(-y^2)\, dy$. The total width of the distribution at time $T$ is

$$\sigma_T^2 = 2a^2 + t = \sigma_1^2 + \mathscr{D}T . \tag{4.12}$$

Unfortunately, this result does not agree with the empirical data. Although the width of the distribution indeed increases with $T$, this increase is achieved by a rounding of the top of the distribution while the slope, on a linear-log plot, of the wings of the distribution remain constant. This prediction clearly disagrees with the observed change in the slope of the wings of the distribution for $1 \leqslant T \leqslant 8$.

These discrepancies can possibly be eliminated if one assumes that the noise $\varepsilon$ in Eq. (4.1) has long-range correlations $\langle \varepsilon_t \varepsilon_{t'} \rangle \sim |t - t'|^{-\gamma}$. Since the analytical solution of the problem is rather complicated, we attempted to solve the problem numerically, assuming for simplicity the Lévy walk [67] type of correlations. We simulate the multiplicative process described by Eq. (4.1), assuming that companies undergo long periods of growth with positive $\varepsilon_t = +\varepsilon$, and long periods of recession with negative

$\varepsilon_t = -\varepsilon$. The durations of these periods $\ell$ we assume to be distributed according to a power-law function

$$p(\ell) \sim \ell^{-\mu}, \quad \mu = \gamma + 2\,. \tag{4.13}$$

These long winning and losing streaks may represent either the general state of the economy of some catastrophic changes in company size, e.g., company merging or splitting, events that do not happen instantaneously, but may, for large corporations, require a long transitional period of several years. In a logarithmic space, the processes of Eq. (4.1) correspond to Lévy walks with unequal time steps: large steps directed toward the origin and small steps directed away from the origin. One can call this unusual type of motion a Lévy walk in a potential field.

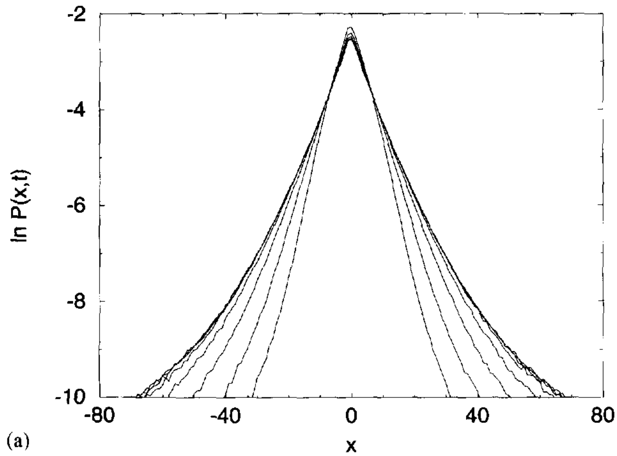
It is well known that classical Lévy walks exhibit superdiffusive behavior when $\mu < 3$ [67]. Our numerical analysis suggests that in this case a Lévy walk in an attractive potential is not confined to the origin but $\sigma_t$ diverges as power law

$$\sigma_t^2 = t^{4-\mu}\,. \tag{4.14}$$

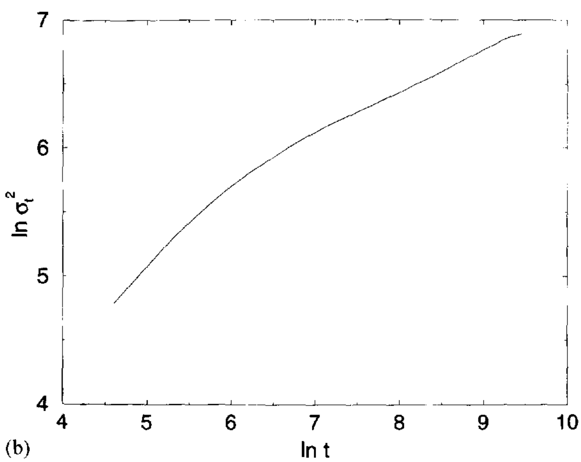
This case clearly does not correspond to our experimental data, since $\sigma_t^2$ grows more slowly than $t$. On the other hand, when $\mu > 3$, Lévy walks are confined by the potential but have very large transient times $t_0$ which diverge as $\mu \to 3 + \varepsilon$. In this case, in the transient regime the distribution of growth rates have a tent-shaped form near the origin, but with power-law wings. Moreover, in this transient regime the slope of the tent shape decreases with time, and $\sigma_t^2$ grows approximately as small power of $t$, thus, exactly reproducing all three unusual features of our experimental data (see Fig. 7). Hence, Lévy-correlated noise may provide a satisfactory explanation of our results. However, additional work is needed to examine other possibilities.

### 4.3. The scaling exponent $\beta$

While the model in the previous section explains Eq. (3.3), it does not predict our finding about the power-law dependence of the standard deviation of growth rates on company size. In this section, we show how a model of management hierarchies can predict Eq. (3.5). In economics, it is generally presumed that the growth of companies is determined by changes in demand and production costs. Since these features are specific to individual markets, it is surprising that a law as simple as Eq. (3.5) governs the growth rate of companies operating in much different markets. While demand and technology vary across markets, virtually all companies have a hierarchical decision structure. One possible explanation for why there is a simple law that governs the growth rate of all manufacturing companies is that the growth process is dominated by properties of management hierarchies [49]. This focus on the technology of management rather then technology of production as a basis for understanding company growth is reminiscent of Lucas' model of the size distribution of companies [41].




(a)




(b)

Fig. 7. (a) Distributions of the Lévy process for $\mu = 3.5$ for several values of $t/100 = 1, 2, 4, 8, 16, 32, 64, 128$, which are simulated as follows. At $t = 0$, the walker is located at the origin $x = 0$. For each time step $\Delta t = 1$, the walker performs one step in a positive or negative direction. The number of consecutive steps $\ell$ performed in the same direction is taken from the power-law distribution $p(\ell) \sim \ell^{-3.5}$. The lengths of steps are equal to 0.9 or $-1.1$ if the current $x$ coordinate of the walker is positive and 1.1 or $-0.9$ otherwise. These unequal steps simulate the effect of the attractive potential. (b) Behavior of $\sigma_t^2$, the width of the distribution obtained using the Lévy process.

At the outset, let us acknowledge a tension between our empirical results and the theoretical model in this section. In the preceding sections, we analyze the scaling properties of the distribution of the logarithmic growth rate $r_1$ and its standard deviation $\sigma_1$. In this section we view companies as consisting of many business units. Since the sales of a company are the sum of the sales of individual units rather than their product, it is more convenient to analyze the standard deviation of the annual company size change rather then the logarithmic growth rate. Let $\Sigma_1(S_0)$ be the standard deviation of end-of-period size for initial size $S_0$. Since $\sigma_1 \sim {S_0}^{-\beta}$ and since $S_1 \equiv S_0 \exp(r_1) \approx S_0 + S_0 r_1$, it follows that $\Sigma_1(S_0) \approx S_0 \sigma_1 \sim {S_0}^{1-\beta}$. We note that $\sigma_1$ must be small for this approximation to hold.

Let us start by assuming that every company, regardless of its size, is made up of similarly sized units. Thus, a company of size $S_0$ is on average made up of $N = S_0/\bar{\xi}$ units, where

$$\bar{\xi} = \frac{1}{N} \sum_{i=1}^{N} \xi_i \,, \tag{4.15}$$

and $\xi_i$ is the size of unit $i$. We further assume that the annual size change $\delta_i$ of each unit follows a bounded distribution with zero mean and variance $\Delta$, which is independent of $S_0$. It is important to notice that throughout this section and the following we consider $\Delta \ll \bar{\xi}^2$, to insure that sizes of units remain positive. Since some divisions after several cycles of growth may shrink almost to zero, while others grow several times, we assume that companies dynamically reorganize themselves so that they begin each period with approximately equal-sized divisions and the inequality $\Delta \ll \bar{\xi}^2$ holds.

If the annual size changes of the different units are independent, then the model is trivial. Using the fact that $\langle \delta_i \rangle = 0$, we obtain

$$\Sigma_1^2(S_0) = N\Delta = S_0 \frac{\Delta}{\bar{\xi}} \sim S_0 \,. \tag{4.16}$$

Using the fact that $\Sigma(S_0) \sim S_0^{1-\beta}$ (see Section 3.4), it follows that $\beta = \frac{1}{2}$ [54].

The much smaller value of $\beta$ that we find indicates the presence of strong positive correlations among a company's units. We can understand this result by considering the tree-like hierarchical organization of a typical company [49]. The head of the tree represents the head of the company, whose policy is passed to the level beneath, and so on, until finally the units in the lowest level take action. These units have again a mean size of $\bar{\xi} = S_0/N$ and annual size changes with zero mean and variance of $\Delta$. Here we assume for simplicity that at every level other than the lowest each node is connected to exactly $z$ units in the next lowest level. Then the number of units $N$ is equal to $z^n$, where $n$ is the number of levels (see Fig. 8).

What are the consequences of this simple model? Let us first assume that the head of the company suggests a policy that could result in changing the size of each unit in the lowest level by an amount $\delta_0$. If this policy is propagated through the hierarchy without any modifications, then it is the same as assuming that all the $\delta_i$'s are identical. This implies that

$$\Sigma_1^2(S_0) = N^2\Delta = S_0^2 \frac{\Delta}{\bar{\xi}^2} \tag{4.17}$$

and we conclude that $\beta = 0$.

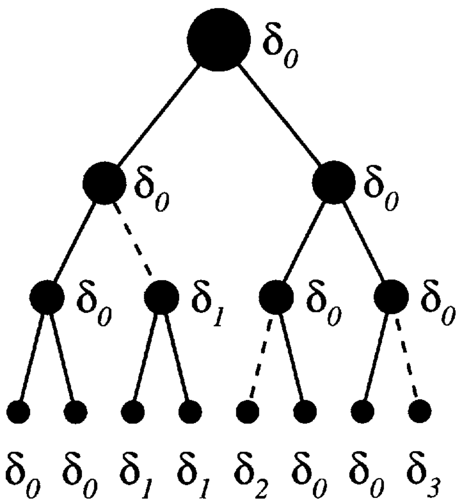



Fig. 8. The hierarchical-tree model of a company. We represent a company as a branching tree. Here, the head of the company makes a decision about the change $\delta_0$ in the size of the lowest level units. That decision is propagated through the tree. However, the decision is only followed with a probability $\Pi$. This is represented in the figure by a full link. With probability $(1 - \Pi)$ a new growth rate is defined. This is represented in the figure by a slashed link. We see that at the lowest level there are clusters of values $\delta_i$ for the changes in size.

Of course, it is not realistic to expect that all decisions in an organization would be perfectly coordinated as if they were all dictated by a single "boss." Hierarchies might be specifically designed to take advantage of information at different levels; and mid-level managers might even be instructed to deviate from decisions made at a higher level if they have information that strongly suggests that an alternative decision would be superior. Another possible explanation for some independence in decision-making is organizational failure, either due to poor communication or disobedience.

To model the intermediate case between $\beta = 0$ and $\beta = \frac{1}{2}$, let us assume that the head of a company makes a decision to change the size of the units of a company by an amount $\delta_0$. We also assume that $\delta_0$, for the set of all companies, has zero mean and variance $\Delta$. Furthermore, we consider that each manager at the nodes of the hierarchical tree follows his supervisor's policy with a probability $\Pi$, while with probability $(1 - \Pi)$ imposes a new *independent* policy. The latter case corresponds to the manager acting as the head of a smaller company made up of the units under his supervision. Hence, the size of the company becomes a random variable with a standard deviation that can be computed either with numerical simulations or using recursion relations among the levels of the tree.

The proposed model is analogous to the expansion modification models used by Li to explain long-range correlations in the DNA sequences [68] and allows a simple analytical solution. In fact, the local production units with numbers $\ell$ and $\ell + k$, where

$k$ is the larger number, are connected to each other through $\log_z k$ levels of company hierarchy. Thus, the correlations among them are equal to $\Pi^{2\log_z k}$, since it is required that $\log_z k$ links going up and $\log_z k$ links going down to connect them. Thus, correlations between production units decay as $k^{2\ln\Pi/\ln z}$. The variance $\Sigma_1^2$ of the total size of $N$ production units is thus

$$\Sigma^2 \sim N^{2+2\ln\Pi/\ln z} \sim S_0^{2+2\ln\Pi/\ln z}\,, \tag{4.18}$$

which implies $\beta = -\ln\Pi/\ln z$. If $-\ln\Pi/\ln z \geqslant \frac{1}{2}$, the units become uncorrelated on large scales and $\Sigma^2$ grows as $S_0$, which implies $\beta = \frac{1}{2}$.

Finally, we can write, for $n \gg 1$, that the hierarchical model leads to

$$\beta = \begin{cases} -\ln\Pi/\ln z & \text{if } \Pi > z^{-1/2}, \\ \frac{1}{2} & \text{if } \Pi < z^{-1/2}. \end{cases} \tag{4.19}$$

Even for small $n$, we find that Eq. (4.19) is a good approximation while for $z = 2$ and $\Pi = 0.87$ we predict $\beta = 0.20$, when we take $n = 3$ the deviation from the predicted value is only 0.03, i.e., about 15%.

Eq. (4.19) is confirmed in the two limiting cases: when $\Pi = 1$ (absolute control) $\beta = 0$, while for all $\Pi < 1/z^{1/2}$, decisions at the upper levels of management have no statistical effect on decisions made at lower levels, and $\beta = \frac{1}{2}$. Moreover, for a given value of $\beta < \frac{1}{2}$ the control level $\Pi$ will be a decreasing function of $z$: $\Pi = z^{-\beta}$, cf. Fig. 9. For example, if we choose the empirical value $\beta \approx 0.15$, then Eq. (4.19) predicts the plausible result $0.9 \geqslant \Pi \geqslant 0.7$ for a range of $z$ in the interval $2 \leqslant z \leqslant 10$.

Our data for $\sigma_T$ suggest that for larger time intervals $\beta$ decreases. Can this be explained within the framework of the hierarchical model? The answer is yes. The decrease in $\beta$ with time suggests that the activity of the company becomes more coordinated on large time scales. It means that the probability $\Pi$ increases with time. This is very plausible, since the information may propagate through the hierarchical structure of the company with finite speed. On small time scales, the activity of the local manager is less coordinated with the general policy of the company headquarters. For example, firing and hiring small numbers of employees may be completely the responsibility of local managers. A major decision, e.g., the firing of a large number of employees, made at the top of the hierarchy is a relatively infrequent event (on a time scale of several years), but when it does occur, it is enforced strictly throughout all levels of the hierarchy.

### 4.4. Combining the two models

We started with two central empirical findings about company growth rates. The model in Section 2 predicts one of those findings (the shape of the distribution) and the model in Section 3 predicts the other (the power-law dependence of the standard deviation of output on company size). This section addresses the relationship between the two models. First, we address concerns that the models might be contradictory and

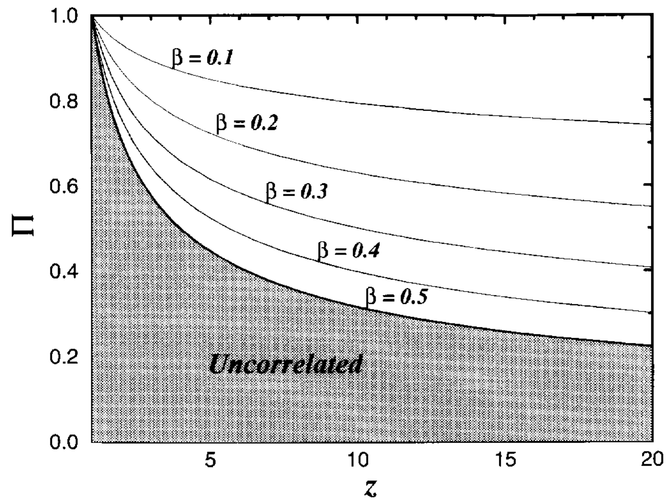


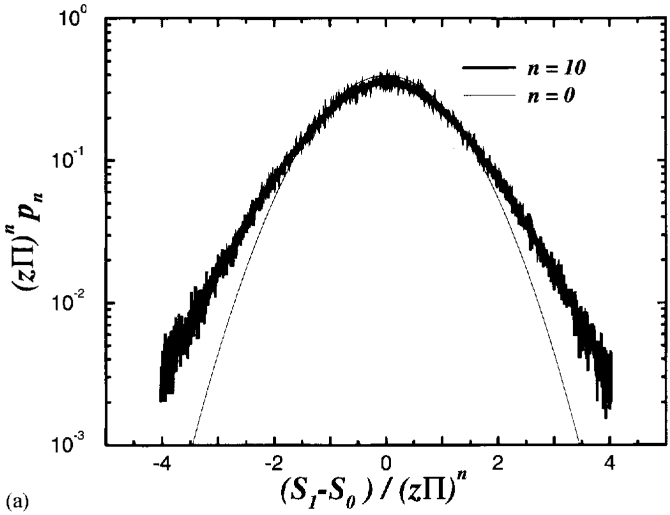

Fig. 9. Phase diagram of the hierarchical-tree model. To each pair of values of $(\Pi, z)$ corresponds a value of $\beta$. We plot the iso-curves corresponding to several values of $\beta$. In the shaded area, marked "Uncorrelated", the model predicts that $\beta = 1/2$, i.e., that the units of the company are uncorrelated. Our empirical data suggests that most companies have values of $\Pi$ and $z$ in between the curves for $\beta = 0.1$ and $\beta = 0.2$.

show that they are not. Then, we show how the models can be combined into a single model that predicts both of our empirical findings.

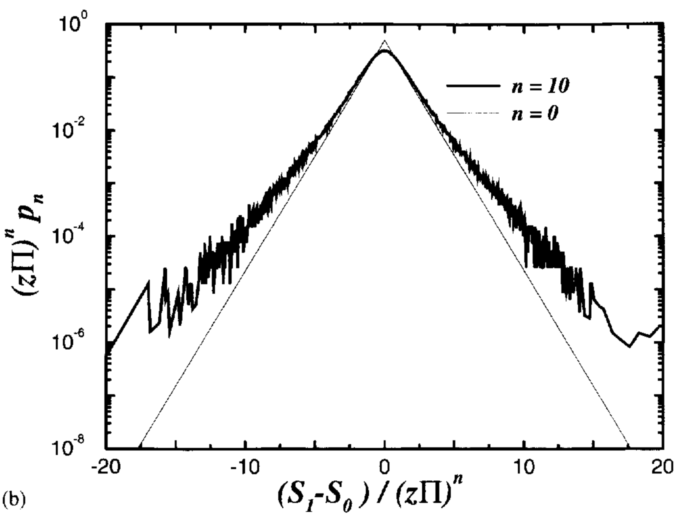
In the tree model, company growth rates are potentially the result of many independent decisions. As a result, one might expect that the central limit theorem would imply a Gaussian distribution of company output. In fact, however, the distribution of outputs is not necessarily Gaussian.

To address the distribution of company output in the tree model, it is necessary to make an assumption about the distribution from which each independent growth decision is drawn. No such assumption is needed to analyze the standard deviation of company growth rates, but is needed to analyze the shape of the distribution [69].

In Fig. 10, we show the distribution of the inputs (i.e., of each independent decision) and the outputs for a tree with $z = 2$, $\Pi = 0.87$, and $n = 10$. We find that for Gaussian distributed inputs, the output is not Gaussian in the tails. This finding is remarkable. First of all, with $z = 2$ and $n = 10$, the company consists of 1024 units. With a probability to disobey of $1 - 0.87 = 0.13$, one would expect $0.13 \times 1024 \approx 133$ of the units to, on average, make independent decisions about their growth rates. Thus, even for non-Gaussian inputs, one can hypothesize that the output is close to Gaussian. Moreover, for Gaussian inputs, the sum of independent Gaussians is itself Gaussian. Thus, for every particular configuration of the disobeying links, the output distribution is Gaussian with variance $m\Delta$, which is a function of this random configuration. However, there are $2^{(z^{n-1} - z)/(z-1)}$ possible configurations of links each of which produce a Gaussian distribution with different integer $m$.




(a)




(b)

Fig. 10. Probability density for the output and input variables in the tree model. Here we have $z = 2$, $\Pi = 0.87$, and $n = 10$. (a) Gaussian distribution of the input. (b) Exponential distribution of the input.

## 5. Conclusions

In summary, we study publicly traded US manufacturing companies from 1974 to 1993. We find that the distribution of the logarithms of the growth rate decays exponentially. Furthermore, we observe that the standard deviation of the distribution of growth rates scales as a power law with the size $S$ of the company, and grows slowly with time $T$. We propose new models that give some insight into these results. We solve the models both numerically and analytically.

The models proposed are quite elementary, and show that simple mechanisms may provide some insight into our findings. Our central results, Eqs. (3.3) and (3.5), constitute a test that any accurate theory of the company must pass, and support the possibility that the scaling laws used to describe complex but inanimate systems comprised of many interacting particles (as occurs in many physical systems) may be usefully extended to describe complex but animate systems comprising of many interacting subsystems (as occurs in economics). Furthermore, the kind of scaling laws found in this study can be viewed as empirical evidence supporting some hypothesis regarding the self-organization of the economy [70].

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