

Rejuvenation in the random energy model

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Abstract. – We show that the Random Energy Model has interesting rejuvenation properties in its frozen phase. Different “susceptibilities” to temperature changes, for the free energy and for other (“magnetic”) observables, can be computed exactly. These susceptibilities diverge at the transition temperature, as $(1 - T/T_c)^{-3}$ for the free energy.

A small temperature change in the low-temperature phase of spin-glasses is able to “rejuvenate” an already aged system [1–3]. More precisely, the a.c. susceptibility of a system aged for a very long time at $T_1 < T_c$ (where T_c is the spin-glass phase transition temperature) and then suddenly cooled at $T_1 - \Delta T$ is, provided ΔT is not too small, very close to the susceptibility of a young system directly cooled from high temperatures to $T_1 - \Delta T$. This “fragility” to temperature changes has been interpreted early on as a signature of “temperature chaos”, that is the fact that the equilibrium states of a disordered system are very different for different temperatures: beyond a certain length scale $\ell_{\Delta T}$ (which diverges for small ΔT), the thermodynamical states become uncorrelated and the overlap between them tends to zero for large system sizes. Such an effect was conjectured in the context of the droplet model of spin-glasses, based on scaling arguments [4, 5], and supported by Migdal-Kadanoff renormalisation group calculations [6]. A similar effect is also predicted for *pinned interfaces* [7], and has recently been checked in careful large-scale simulations and analytical arguments in the case of the 1 + 1 directed polymer in random media [8]. However, temperature chaos has been recently disproven in the mean-field SK model [9], and not been found either in numerical simulations of the 3D Edwards-Anderson spin-glass model [10, 11]. This might be due to the fact that the length $\ell_{\Delta T}$ (if it exists) involves a large numerical prefactor and is therefore larger than numerically accessible sizes. It could also be that although the length scale $\ell_{\Delta T}$ plays a relevant role in the overlap between the states at T_1 and $T_1 - \Delta T$, this overlap does tend very slowly to zero for large sizes, as in [8]. Finally, the “temperature chaos” interpretation of the rejuvenation effect has to be compatible with the simultaneous *memory* that one observes experimentally. A scenario for this was recently proposed in [12].

Another line of thought to explain rejuvenation and memory in spin-glasses is based on “hierarchical” energy landscape pictures [13, 14]. Lower temperatures reveal finer details (rejuvenation) while large-scale jumps are frozen out (memory) [2, 3, 15]. Numerical simulations of the dynamics on Parisi’s tree [16] or in the Sinai potential [17] indeed confirm that these effects exist in the absence of “true” chaos. Even a simple two-level system does actually lead to some rejuvenation when the temperature is of the order of the energy difference between the levels —simply because the relative Boltzmann weight changes when the temperature is

changed. In this paper, we want to study in details the temperature rejuvenation effect in the glassy phase of the Random Energy Model (REM), for which a number of exact results are known [18]. This low-temperature phase is described, in the replica language, by a “one-step” replica-symmetry-breaking scheme [19]. This model is expected to be in the same universality class (with possibly minor corrections) as a large number of other models [20], such as the p -spin models (advocated to be good models for glasses), the Bernasconi model, the directed polymer (or random manifold) in high dimensions [21,22], the unbinding transition of heteropolymers [23] and the problem of a single particle in a logarithmically correlated random potential [24]. Interestingly, a dynamical version of the REM naturally leads to aging dynamics [15,25,26].

Here, we want to compute exactly the “rejuvenation susceptibility” of different observables to small temperature changes. Even if the model is not “chaotic”, it reveals a number of interesting features that may be relevant to the present discussion. For example, the “susceptibility” to temperature changes diverges when the temperature tends to the glass temperature of the model. Experimental consequences are discussed in the conclusion.

It is well known that the low-temperature phase of the REM is equivalent to that of the “trap” model [27,28], where M energy states ϵ_i , $i = 1, \dots, M$ are chosen with an exponential probability distribution:

$$P(\epsilon) = \frac{1}{T_c} \exp \left[-\frac{|\epsilon|}{T_c} \right]. \quad (1)$$

Note that ϵ is chosen to be negative. The partition function for this model is simply $Z(T) = \sum_{i=1}^M z_i$ with $z_i = \exp[|\epsilon_i|/T]$. This model undergoes a phase transition at $T = T_c$, where the partition function “localizes” on a few states. More precisely, in the limit $M \rightarrow \infty$, the Boltzmann weights of a finite number of states add up to a finite fraction of the partition function for $T < T_c$ [27,28]. Aging is the dynamical counterpart of this localization effect: most of the elapsed time is spent by the system in the deepest available well [15].

As a first definition of the susceptibility to temperature changes, we study, following Fisher and Huse [7], the correlation of the free-energy fluctuations for two different temperatures. More precisely, we write

$$C_F(T_1, T_2) = \frac{\overline{(\ell_1 - \bar{\ell}_1)(\ell_2 - \bar{\ell}_2)}}{((\ell_1 - \bar{\ell}_1)^2(\ell_2 - \bar{\ell}_2)^2)^{1/2}}, \quad (2)$$

where ℓ_1 stands for $\log Z(T_1)$ and the overline means that we average over the distribution of the energies ϵ . When $T_1 = T_2(1 + \varepsilon)$ we expect that

$$C_F(T_1, T_2) = 1 - \kappa_F \varepsilon^2 \quad (\varepsilon \rightarrow 0), \quad (3)$$

where κ_F defines the susceptibility to temperature changes. The calculation of this quantity starts with Derrida’s representation of $\log Z$ [18]:

$$\log Z = \int_0^\infty dt \frac{\exp[-t] - \exp[-tZ]}{t} = \lim_{b \rightarrow 0^+} \int_0^\infty dt t^{b-1} (\exp[-t] - \exp[-tZ]), \quad (4)$$

where b has been introduced to ensure convergence of intermediate calculation steps. The average over ϵ then involves

$$\overline{\exp[-tZ]} = (\overline{\exp[-tz]})^M = (1 - (1 - \overline{\exp[-tz]})^M). \quad (5)$$

For large M , only the vicinity of $t = 0$ will therefore be of importance. Using the fact that the random variables z are distributed with a power law tail $\mu z^{-1-\mu}$ with $\mu = T/T_c$, one finds that

$$1 - \overline{\exp[-tz]} \sim_{t \rightarrow 0} \Gamma(1 - \mu)t^\mu. \quad (6)$$

Using the last result in the previous two equations finally leads to the following result for the free energy (in units of T_c):

$$-\overline{\log Z} = -\log M - \gamma(1 - \mu) - \log \Gamma(1 - \mu), \quad (7)$$

where γ is Euler's constant. The first term comes from the fact that, when M is large, the smallest energy drawn from an exponential distribution behaves as $-T_c \log M$, plus order one (random) corrections. Therefore, all the fluctuations involved in the calculation of C_F will be of order 1. The calculation of the average of the product of $\log Z$ for two different temperatures is a little more involved. At an intermediate level of the computation, one finds, to order b^0 ,

$$\begin{aligned} \overline{\ell_1 \ell_2} = & \Gamma^2(b) - \Gamma(b) \left[\frac{\Gamma(b/\mu_1)}{M^{b/\mu_1} \mu_1 \Gamma^{b/\mu_1}(1 - \mu_1)} + 1 \longrightarrow 2 \right] + \\ & + \frac{\Gamma(b\beta)}{\mu_1 b M^{b\beta}} \left[\frac{1 - b^2 \beta \mathcal{F}(\mu_1, \alpha)}{\Gamma^{b\beta}(1 - \mu_1)} + \alpha \frac{1 - b^2 \beta \mathcal{F}(\mu_2, 1/\alpha)}{\Gamma^{b\beta}(1 - \mu_2)} \right] \end{aligned} \quad (8)$$

with $\alpha = 1 + \varepsilon$, $\beta = (1 + \alpha)/\mu_1$ and

$$\mathcal{F}(\mu, \alpha) = \int_0^1 \frac{dv}{v} \log \left[1 + \frac{\mu}{\Gamma(1 - \mu)} \int_0^\infty du \frac{\exp[-u] - \exp[-u^\alpha v]}{u^{1+\mu}} \right]. \quad (9)$$

Expanding the previous result to order ε^2 and rearranging the terms finally leads to $\kappa_F(\mu_1)$, given in the appendix. Its dependence in μ_1 is given in fig. 1. Of main interest is its behaviour for small temperatures and close to the transition point. For $\mu_1 \rightarrow 0$, one finds that κ_F goes to zero as

$$\kappa_F(\mu_1) \sim 0.905558 \cdots \mu_1^2. \quad (10)$$

For small temperatures, only the ground state contributes to the free energy both for T_1 and T_2 . Hence, one indeed expects the sample-to-sample fluctuations to be strongly correlated. For $\mu_1 \rightarrow 1$, on the contrary, one finds that κ_F diverges:

$$\kappa_F(\mu_1) \sim \frac{3(4 \log 2 - 1)}{\pi^2 (1 - \mu_1)^3} \sim \frac{0.538802 \cdots}{(1 - \mu_1)^3}, \quad (11)$$

indicating that close to the ‘‘delocalisation’’ transition, the system tends to occupy rather different states when the temperature is slightly changed. Finally, the correlation of free-energy fluctuations between $T_1 > T_c$ and $T_2 < T_c$ vanishes as the size of the system tends to infinity.

Therefore, except right at the transition, there is no ‘‘strong chaos’’ in the REM, which would imply that the susceptibility κ_F diverges with the size of the system. However, small temperature changes do lead to noticeable changes in the physical observables. A perhaps more direct way to see this is to assign to each state i a certain observable \mathcal{O}_i , independent of the energy of this state. This can be, for example, the position of the state in space if the

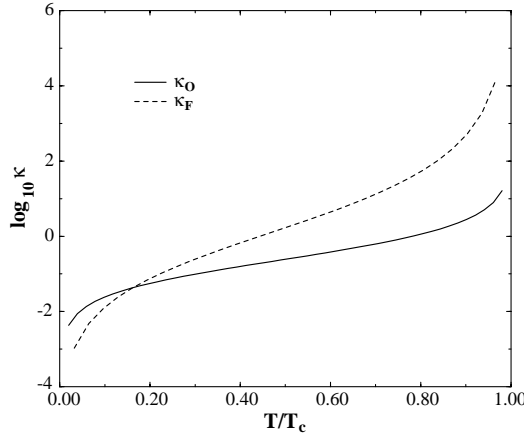


Fig. 1 – Plot of the rejuvenation susceptibilities κ_{O} (straight line) and κ_{F} (dotted line) in semi-log scale, as a function of $\mu = T/T_c$. Note that both quantities vanish at $T = 0$ and diverge at $T = T_c$, with different exponents.

model describes (for example) the metastable states of a pinned interface, or the magnetisation of a state for spin system. We will assume for simplicity that $\sum_i \mathcal{O}_i = 0$. For a given set of random energies ϵ_i , the thermodynamical value of the observable is

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \sum_{i=1}^M z_i \mathcal{O}_i. \quad (12)$$

The average over disorder of this quantity is always zero. However, a fundamental difference arises between the case $T > T_c$ and $T < T_c$. In the former case and for large systems, $\langle \mathcal{O} \rangle$ tends to zero. This is related to the fact that the partition function is more or less evenly spread out on all states. On the contrary, for $T < T_c$, $\langle \mathcal{O} \rangle$ for a given system is a finite random quantity. Its variance is (for large M) given by

$$\overline{\langle \mathcal{O} \rangle^2} = (1 - \mu) \sum_i \mathcal{O}_i^2, \quad \mu = \frac{T}{T_c}. \quad (13)$$

Take, for example, the case where the observable $\mathcal{O}_i = x_i$ is the position of a particle in a box of size $2L$, $x_i = L(1 - 2i/M)$. In this case, the typical average position of the particle is $\langle x \rangle \sim \sqrt{1 - \mu}L$ which scales with the total size of the system, compared to $\langle x \rangle \sim L^\zeta$ with $\zeta < 1$ for $T > T_c$. Quenching the temperature from above T_c therefore induces a complete rearrangement of the equilibrium properties of the system which occurs in a slow, aging way.

Now, let us discuss how this observable changes when the temperature is slightly changed *within* the glass phase. The calculation proceeds much as above, or as in [27]. For small temperature shifts $\varepsilon \rightarrow 0$, we find

$$\overline{(\langle \mathcal{O} \rangle_1 - \langle \mathcal{O} \rangle_2)^2} = \kappa_{\text{O}}(\mu_1) \varepsilon^2 \left(\sum_i \mathcal{O}_i^2 \right), \quad (14)$$

where the rejuvenation susceptibility is given by a very lengthy expression. This quantity is

plotted in fig. 1. Its small temperature behaviour is given by

$$\kappa_{\text{O}}(\mu_1) \sim \frac{12 + \pi^2}{18} \mu_1 = 1.21497802 \cdots \mu_1, \quad (15)$$

where its divergence for $\mu_1 \rightarrow 1$ is given by

$$\kappa_{\text{O}}(\mu_1) \sim \frac{1}{3(1 - \mu_1)}. \quad (16)$$

Therefore, we again find a divergence of the rejuvenation susceptibility near T_c . For $T > T_c$, we find that κ_{O} vanishes as a (T -dependent) power of M .

We then find a very interesting situation: eq. (14) tells us that when the temperature is slightly changed, the mean position of the particle (say) has to evolve by an amount of order $\sqrt{\kappa \varepsilon L}$ that is *proportional to the size of the system* (and with a diverging amplitude when $T \rightarrow T_c$). In this sense, rejuvenation is strong since a small temperature change will induce a rather large response of the system. However, since the number of states occupied by the particle remains finite in the whole glassy phase $T < T_c$, the probability to find the system in the same state at T_1 and T_2 remains finite in the limit of large systems [29]. This probability is directly related to κ_{O} , and reads

$$P_{12} = 1 - \frac{\mu_1 + \mu_2}{2} - \kappa_{\text{O}} \varepsilon^2. \quad (17)$$

(Note that $P_{11} = 1 - \mu_1$ as it should [27, 28].)

In summary, we have shown that the Random Energy Model has interesting rejuvenation properties in its frozen phase. Different “susceptibilities” to temperature changes, for the free energy and for other (magnetic) observables, can be computed exactly. These susceptibilities diverge at the transition temperature with different exponents. Since the REM seems to be relevant to many physical situations, the mechanism found here is probably of broad interest. However, the coexistence of rejuvenation and memory seen in the spin-glass experiments cannot be accounted for by the simplest version of the REM, because the evolution at T_2 will have a significant influence on the properties measured at T_1 after reheating. One can generalize the REM along the lines of [30] by allowing a hierarchy of phase transitions $T_{c,n}$. As argued in [15] and numerically demonstrated in [16], each crossing of a phase transition $T_{c,n}$ induces a strong rejuvenation signal (much as calculated here) with a slow aging dynamics for $T < T_{c,n}$ and a fast return to equilibrium for $T > T_{c,n}$, accounting for the memory effect.

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APPENDIX

Here we give the explicit form of the free-energy susceptibility $\kappa_{\text{F}}(\mu)$. We find

$$\begin{aligned}
(2\pi^2(\mu-1)^2(\mu+1)^2)\kappa_F(\mu) &= 6(1-2\ell_2+2\ell_2\psi'[1-\mu]\mu^5) - \\
&- 12\mu^3(-1+\ell_2\psi'[1-\mu]) + \mu^2(\pi^2-2(-9+3\psi'[1-\mu]+6\psi[-\mu])) - \\
&- 3\psi[-\mu]^2 + 12\ell_2 - 18\ell_2\psi[-\mu] + 6\ell_2\psi^2[-\mu] + 9\ell_2^2 + 2\ell_2^3 - 6\ell_2^2\psi[-\mu]) + \\
&+ 12\mu(1-\ell_2(3+\ell_2) + (-1+2\ell_2)\psi[-\mu]) + 2\mu^4 \cdot \\
&\cdot (3\psi'[1-\mu] - 6\ell_2(1+\ell_2)\psi[-\mu] + \ell_2^2(3+2\ell_2) + 3(-1+2\ell_2)\psi^2[-\mu]),
\end{aligned}$$

where $\ell_2 \equiv \log 2$ and ψ is the logarithmic derivative of the Γ -function. The susceptibility κ_O is given by a similar, although even more involved, expression.

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